PP44842

MIHÁLY BENCZE - ROMANIA

In all triangles ABC holds:

$$\sum_{cyc} \frac{a(b+c-a)}{(a-b+c)(a+b-c)} \geq 3$$

Solution by Daniel Sitaru, Claudia Nănuţi.

$$\sum_{cyc} \frac{a(b+c-a)}{(a-b+c)(a+b-c)} = \sum_{cyc} \frac{a(2s-2a)}{(2s-2b)(2s-2c)} =$$

$$= \frac{2}{4} \sum_{cyc} \frac{a(s-a)}{(s-b)(s-c)} = \frac{1}{2(s-a)(s-b)(s-c)} \sum_{cyc} a(s-a)^2 =$$

$$= \frac{s}{2s(s-a)(s-b)(s-c)} \sum_{cyc} a(s^2 - 2sa + a^2) =$$

$$= \frac{s}{2F^2} \left(s \sum_{cyc} a - 2s \sum_{cyc} a^2 + \sum_{cyc} a^3 \right) =$$

$$= \frac{s}{2F^2} (s^2 \cdot 2s - 2s(2s^2 - 2r^2 - 8Rr) + 2s^3 - 6sr^2 - 12Rrs)$$

$$= \frac{s^2}{2r^2 s^2} (2s^2 - 4s^2 + 4r^2 + 16Rr + 2s^2 - 6r^2 - 12Rr) =$$

$$= \frac{1}{2r^2} (4Rr - 2r^2) = \frac{2R-r}{r} \overset{\text{Euler}}{\geq}$$

$$\geq \frac{2 \cdot 2r - r}{r} = \frac{3r}{r} = 3$$

Equality holds for $a = b = c$.

□

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