PP44862

MIHÁLY BENCZE - ROMANIA

If $a > 1$ and $n \in \mathbb{N}$ compute:

$$\int_{\frac{1}{a}}^{a} \frac{x^n \arctan x}{x^{2n+2} + x^{n+1} + 1} dx$$

Solution by Daniel Sitaru and Claudia Nănuţi.

$$\text{Let be } y = \frac{1}{x} \Rightarrow x = \frac{1}{y} \Rightarrow dx = \frac{-1}{y^2} dy$$

$$x = \frac{1}{a} \Rightarrow y = a$$

$$x = a \Rightarrow y = \frac{1}{a}$$

$$\Omega = \int_{\frac{1}{a}}^{a} \frac{x^n \arctan x}{x^{2n+2} + x^{n+1} + 1} dx = \int_{a}^{\frac{1}{a}} \frac{\frac{1}{y^n} \arctan \frac{1}{y}}{\frac{1}{y^{2n+2}} + \frac{1}{y^{n+1}} + 1} \Big(-\frac{1}{y^2} \Big) dy =$$

$$= \int_{\frac{1}{a}}^{a} \frac{\frac{1}{y^{n+2}} \arctan \frac{1}{y}}{\frac{y^{2n+2} + y^{n+1} + 1}{y^{2n+2}}} \cdot dy = \int_{\frac{1}{a}}^{a} \frac{y^n \arctan \frac{1}{y}}{y^{2n+2} + y^{n+1} + 1} dy =$$

$$\Omega = \int_{\frac{1}{a}}^{a} \frac{x^n \arctan x}{x^{2n+2} + x^{n+1} + 1} dx \tag{1}$$

$$\Omega = \int_{\frac{1}{a}}^{a} \frac{x^n \arctan \frac{1}{x}}{x^{2n+2} + x^{n+1} + 1} dx \tag{2}$$

By adding (1); (2):

$$2\Omega = \int_{\frac{1}{a}}^{a} \frac{x^n (\arctan x + \arctan \frac{1}{x})}{x^{2n+2} + x^{n+1} + 1} dx$$

$$\Omega = \frac{1}{2} \cdot \frac{\pi}{2} \int_{\frac{1}{a}}^{a} \frac{x^n}{(x^{n+1} + \frac{1}{2})^2 + \frac{3}{4}} dx$$

$$\Omega = \frac{\pi}{4(n+1)} \int_{\frac{1}{a}}^{a} \frac{(x^{n+1} + \frac{1}{2})'}{(x^{n+1} + \frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} dx$$

$$\Omega = \frac{\pi}{4(n+1)} \cdot \arctan \left(\frac{x^{n+1} + \frac{1}{2}}{\frac{\sqrt{3}}{2}} \right) \Bigg|_{\frac{1}{a}}^{a}$$

$$\Omega = \frac{\pi}{4(n+1)} \Big(\arctan \Big(\frac{2a^{n+1} + 1}{\sqrt{3}} \Big) - \arctan \Big(\frac{2 + a^{n+1}}{\sqrt{3}} \Big) \Big)$$

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