PP44899

MIHÁLY BENCZE - ROMANIA

In all triangles ABC holds:

$$3 \geq \sum_{cyc} \frac{h_a + h_b}{r_a + r_b} \geq \frac{6r}{R}$$

Solution by Daniel Sitaru and Claudia Nănuţi.
Lemma.
In ΔABC the following relationship holds:

$$\sum_{cyc} (a+b)(s-a)(s-b) = 4sr(R+r)$$

Proof.

$$\sum_{cyc} (a+b)(s-a)(s-b) = \sum_{cyc} (a+b)(s^2 - s(a+b) + ab) =$$

$$= \sum_{cyc} (2s-c)(s^2 - s(2s-c) + ab) =$$

$$= \sum_{cyc} (2s-c)(s^2 - 2s^2 + sc + ab) =$$

$$= \sum_{cyc} (2s-c)(sc + ab - s^2) =$$

$$= \sum_{cyc} (2s^2c + 2sab - 2s^3 - sc^2 - abc + s^2c) =$$

$$= 3s^2 \cdot 2s + 2s \sum_{cyc} ab - 6s^3 - s \sum_{cyc} c^2 - 3abc =$$

$$= 2s(s^2 + r^2 + 4Rr) - s \cdot 2(s^2r^2 - 4Rr) - 12Rrs =$$

$$= 2s^3 + 2sr^2 + 8Rrs - 2s^3 + 2sr^2 + 8Rrs - 12Rrs =$$

$$= 4sr^2 + 4Rrs = 4sr(R+r)$$

□

Back to the problem:

$$\sum_{cyc} \frac{h_a + h_b}{r_a + r_b} = \sum_{cyc} \frac{\frac{2F}{a} + \frac{2F}{b}}{\frac{F}{s-a} + \frac{F}{s-b}} =$$

$$= \frac{2F}{F} \sum_{cyc} \frac{\frac{1}{a} + \frac{1}{b}}{\frac{1}{s-a} + \frac{1}{s-b}} = 2 \sum_{cyc} \frac{\frac{a+b}{ab}}{\frac{s-a+s-b}{(s-a)(s-b)}} =$$

$$= \frac{2}{abc} \sum_{cyc} (a+b)(s-a)(s-b) \overset{\text{Lemma}}{=}$$

$$=\frac{2}{4Rrs}\cdot 4sr(R+r)=\frac{2(R+r)}{R}$$

$$\sum_{cyc}\frac{h_a+h_b}{r_a+r_b}\geq\frac{6r}{R}\Leftrightarrow\frac{2(R+r)}{R}\geq\frac{6r}{R}\Leftrightarrow$$

$$\Leftrightarrow 2R+2r\geq 6r\Leftrightarrow 2R\geq 4r$$

$$R\geq 2r\ \text{(Euler)}$$

$$\sum_{cyc}\frac{h_a+h_b}{r_a+r_b}\leq 3\Leftrightarrow\frac{2(R+r)}{R}\leq 3\Leftrightarrow$$

$$\Leftrightarrow 2R+2r\leq 3R\Leftrightarrow R\geq 2r\ \text{(Euler)}$$

Equality holds for $a=b=c$.

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