PP44900

MIHÁLY BENCZE - ROMANIA

In all triangles ABC holds:

$$\frac{96r^2}{s^2+r^2+2Rr} \le \sum_{cyc} \frac{(r_a+r_b)(r_b+r_c)}{(h_a+h_b)(h_b+h_c)} \le \frac{12R^3}{r(s^2+r^2+2Rr)}$$

Solution by Daniel Sitaru, Claudia Nănuţi.
As we proved at the problem PP44902:

$$\sum_{cyc} \frac{(r_a+r_b)(r_b+r_c)}{(h_a+h_b)(h_b+h_c)} = \frac{8R^2(R+r)}{r(s^2+r^2+2Rr)}$$

Remains to prove that:

$$\frac{96r^2}{s^2+r^2+2Rr} \le \frac{8R^2(R+r)}{r(s^2+r^2+2Rr)} \le \frac{12R^3}{r(s^2+r^2+2Rr)}$$

$$24r^2 \le \frac{2R^2(R+r)}{r} \le \frac{3R^3}{r}$$

$$24r^3 \le 2R^2(R+r) \le 3R^3 \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 2R^2(R+r) \ge 24r^3 \\ 2R^2(R+r) \le 3R^3 \end{cases}$$

$$2R^2(R+r) \ge 24r^3 \Leftrightarrow R^3+R^2r \ge 12r^3 \text{ (to prove)}$$

$$R^3+R^2r \overset{\text{EULER}}{\ge} (2r)^3+(2r)^2r = 8r^3+4r^3 = 12r^3$$

$$\Leftrightarrow 3R \ge 2R+2r \Leftrightarrow R \ge 2r \text{ (EULER)}$$

Equality holds for $a=b=c$. □

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