PP44901

MIHÁLY BENCZE - ROMANIA

In all triangles ABC holds:

$$\frac{6r}{R} \leq \sum_{cyc} \frac{(h_a+h_b)(h_b+h_c)}{(r_a+r_b)(r_b+r_c)} \leq \frac{R^2+3Rr+2r^2}{R^2}$$

Solution by Daniel Sitaru and Claudia Nănuţi.
As we proved at the problem PP44902.

$$\sum_{cyc} \frac{(h_a+h_b)(h_b+h_c)}{(r_a+r_b)(r_b+r_c)} = \frac{s^2+8Rr+5r^2}{4R^2}$$

We must prove that:

$$\frac{6r}{R} \leq \frac{s^2+8Rr+5r^2}{4R^2} \leq \frac{R^2+3Rr+2r^2}{R^2} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 24Rr \leq s^2+8Rr+5r^2 \\ s^2+8Rr+5r^2 \leq 4R^2+12Rr+8r^2 \end{cases}$$

$$\Leftrightarrow \begin{cases} s^2 \geq 16Rr-5r^2 \text{ (GERRETSEN)} \\ s^2 \leq 4R^2+4Rr+3r^2 \text{ (GERRETSEN)} \end{cases}$$

Equality holds for $a=b=c$.

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