PP44902

MIHÁLY BENCZE - ROMANIA

In all triangles ABC holds:

$$1.\ \sum_{cyc} \frac{(h_a+h_b)(h_b+h_c)}{(r_a+r_b)(r_b+r_c)} = \frac{s^2+8Rr+5r^2}{4R^2}$$

$$2.\ \sum_{cyc} \frac{(r_a+r_b)(r_b+r_c)}{(h_a+h_b)(h_b+h_c)} = \frac{8R^2(R+r)}{r(s^2+r^2+2Rr)}$$

Solution by Daniel Sitaru and Claudia Nănuţi.

Lemma 1.

In all triangles ABC holds:

$$\sum_{cyc}(s-b)(a+b)(b+c) = s(s^2+5r^2+8Rr)$$

Proof.

$$\sum_{cyc}(s-b)(a+b)(b+c) = \sum_{cyc}(s-b)(2s-c)(2s-a) =$$

$$= \sum_{cyc}(s-b)(4s^2-2s(a+c)+ac) =$$

$$= \sum_{cyc}(s-b)(4s^2-2s(2s-b)+ac) =$$

$$= \sum_{cyc}(s-b)(4s^2-4s^2+2sb+ac) =$$

$$= \sum_{cyc}(s-b)(2sb+ac) =$$

$$= \sum_{cyc}(2s^2b+sac-2sb^2-abc) =$$

$$= 2s^2\cdot 2s + s\sum_{cyc} ac - 2s\sum_{cyc} b^2 - 3abc =$$

$$= 4s^3+s(s^2+r^2+4Rr)-2s\cdot 2(s^2-r^2-4Rr)-3abc =$$
$$= 4s^3+s^3+sr^2+4Rrs-4s^3+4sr^2+16Rrs-12Rrs =$$
$$= s^3+5sr^2+8Rrs = s(s^2+5r^2+8Rr)$$

□

Lemma 2.

In all triangles ABC holds:

$$\sum_{cyc}(s-a)(s-c)(a+c) = 4sr(R+r)$$

Proof.

$$\sum_{cyc}(s-a)(s-c)(a+c)=$$
$$=\sum_{cyc}(s^2-s(a+c)+ac)(2s-b)=$$
$$=\sum_{cyc}(s^2-s(2s-b)+ac)(2s-b)=$$
$$=\sum_{cyc}(s^2-2s^2+sb+ac)(2s-b)=$$
$$=\sum_{cyc}(sb+ac-s^2)(2s-b)=$$
$$=\sum_{cyc}(2s^2b-sb^2+2sac-abc-2s^3+s^2b)=$$
$$=3s^2\cdot 2s-s\sum_{cyc}b^2+2s\sum_{cyc}ac-3abc-6s^3=$$
$$=6s^2-s\cdot 2(s^2-r^2-4Rr)+2s(s^2+r^2+4Rr)-3abc-6s^3=$$
$$=-2s^3+2sr^2+8Rrs+2s^3+2sr^2+8Rrs-12Rrs=$$
$$=4sr^2+4Rrs=$$
$$=4sr(R+r)=$$

□

Back to the main problem 1:

$$\sum_{cyc}\frac{(h_a+h_b)(h_b+h_c)}{(r_a+r_b)(r_b+r_c)}=\sum_{cyc}\frac{(\frac{2F}{a}+\frac{2F}{b})(\frac{2F}{b}+\frac{2F}{c})}{(\frac{F}{s-a}+\frac{F}{s-b})(\frac{F}{s-b}+\frac{F}{s-c})}=$$
$$=\frac{4F^2}{F^2}\sum_{cyc}\frac{(\frac{1}{a}+\frac{1}{b})(\frac{1}{b}+\frac{1}{c})}{(\frac{1}{s-a}+\frac{1}{s-b})(\frac{1}{s-b}+\frac{1}{s-c})}=$$
$$=4\sum_{cyc}\frac{\frac{a+b}{ab}\cdot\frac{b+c}{bc}}{\frac{s-a+s-b}{(s-a)(s-b)}\cdot\frac{s-c+s-b}{(s-b)(s-c)}}=$$
$$=4\sum_{cyc}\frac{(a+b)(b+c)}{ab^2c}\cdot\frac{(s-a)(s-b)^2(s-c)}{ca}=$$
$$=\frac{4(s-a)(s-b)(s-c)}{(abc)^2}\sum_{cyc}(s-a)(a+b)(b+c)=$$
$$\overset{\text{Lemma 1}}{=}\frac{4s(s-a)(s-b)(s-c)}{s(abc)^2}\cdot s(s^2+5r^2+8Rr)=$$
$$=\frac{4\cdot F^2}{16R^2F^2}\cdot(s^2+5r^2+8Rr)=$$
$$=\frac{s^2+5r^2+8Rr}{4R^2}$$

Back to the main problem 2:

$$\sum_{cyc}\frac{(r_a+r_b)(r_b+r_c)}{(h_a+h_b)(h_b+h_c)}=$$

$$= \sum_{cyc} \frac{(\frac{F}{s-a} + \frac{F}{s-b})(\frac{F}{s-b} + \frac{F}{s-c})}{(\frac{2F}{a} + \frac{2F}{b})(\frac{2F}{b} + \frac{2F}{c})} =$$

$$= \frac{F^2}{4F^2} \sum_{cyc} \frac{\frac{s-b+s-a}{(s-a)(s-b)} \cdot \frac{s-c+s-b}{(s-b)(s-c)}}{\frac{a+b}{ab} \cdot \frac{b+c}{bc}} =$$

$$= \frac{1}{4} \sum_{cyc} \frac{\frac{ac}{(s-a)(s-b)^2(s-c)}}{\frac{(a+b)(b+c)}{ab^2c}} =$$

$$= \frac{(abc)^2}{4(s-a)(s-b)(s-c)} \sum_{cyc} \frac{1}{(s-b)(a+b)(b+c)} =$$

$$= \frac{(abc)^2}{4((s-a)(s-b)(s-c))^2} \cdot \frac{1}{(a+b)(b+c)(c+a)} \cdot \sum_{cyc} (s-a)(s-c)(a+c) =$$

$$\overset{\text{Lemma 2}}{=} \frac{(abc)^2 s^2}{4F^4} \cdot \frac{1}{2s(s^2+r^2+2Rr)} \cdot 4sr(R+r) =$$

$$= \frac{16R^2F^2s^2 \cdot 4sr(R+r)}{8sF^4(s^2+r^2+2Rr)} =$$

$$= \frac{2R^2 \cdot 4s^2r(R+r)}{F^2(s^2+r^2+2Rr)} = \frac{8s^2R^2r(R+r)}{r^2s^2(s^2+r^2+2Rr)} =$$

$$= \frac{8R^2(R+r)}{r(s^2+r^2+2Rr)}$$

□

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