PP44906

MIHÁLY BENCZE - ROMANIA

In all triangles ABC holds:

$$1.\ \sum_{cyc}\frac{r_a+r_b}{h_a+h_b}=\frac{R(s^2+5r^2+8Rr)}{r(s^2+r^2+2Rr)}$$

$$2.\ \sum_{cyc}\frac{h_a+h_b}{r_a+r_b}=\frac{2(R+r)}{R}$$

Solution by Daniel Sitaru and Claudia Nănuţi.
We will use the results obtained at the problems: PP44907 and PP44902:

$$\sum_{cyc}(s-c)(b+c)(c+a)=s(s^2+5r^2+8Rr)\ \text{(PP44907)}$$

$$\sum_{cyc}(s-a)(s-b)(a+b)=4sr(R+r)\ \text{(PP44902)}$$

$$1.\ \sum_{cyc}\frac{r_a+r_b}{h_a+h_b}=\sum_{cyc}\frac{\frac{F}{s-a}+\frac{F}{s-b}}{\frac{2F}{a}+\frac{2F}{b}}=\frac{F}{2F}\sum_{cyc}\frac{\frac{1}{s-a}+\frac{1}{s-b}}{\frac{1}{a}+\frac{1}{b}}=$$

$$=\frac{1}{2}\sum_{cyc}\frac{\frac{s-b+s-a}{(s-a)(s-b)}}{\frac{a+b}{ab}}=\frac{abc}{2}\sum_{cyc}\frac{1}{(a+b)(s-a)(s-b)}=$$

$$=\frac{abc}{2}\cdot\frac{1}{(a+b)(b+c)(c+a)(s-a)(s-b)(s-c)}\sum_{cyc}(s-c)(b+c)(c+a)=$$

$$\overset{\text{PP44907}}{=}\frac{4RFs}{2(a+b)(b+c)(c+a)F^2}\cdot s(s^2+5r^2+8Rr)=$$

$$=\frac{2Rs}{rs\cdot 2s(s^2+r^2+2Rr)}\cdot s(s^2+5r^2+8Rr)=$$

$$=\frac{R(s^2+5r^2+8Rr)}{r(s^2+r^2+2Rr)}$$

$$2.\ \sum_{cyc}\frac{h_a+h_b}{r_a+r_b}=\sum_{cyc}\frac{\frac{2F}{a}+\frac{2F}{b}}{\frac{F}{s-a}+\frac{F}{s-b}}=$$

$$=\frac{2F}{F}\sum_{cyc}\frac{\frac{a+b}{ab}}{\frac{s-a+s-b}{(s-a)(s-b)}}=\frac{2}{abc}\sum_{cyc}(s-a)(s-b)(a+b)=$$

$$\overset{\text{PP44902}}{=}\frac{2}{4Rrs}\cdot 4sr(R+r)=\frac{2(R+r)}{R}$$

$\square$

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