PP44943

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In all triangles ABC holds:

$$\frac{9r}{2R}\sqrt{\frac{r}{6R}} \leq \sin A \sin B \sin C \leq \frac{r(4R+r)}{2R^2\sqrt{3}}$$

Solution by Daniel Sitaru and Claudia Nănuţi.

$$\sin A \sin B \sin C = \frac{a}{2R} \cdot \frac{b}{2R} \cdot \frac{c}{2R} = \frac{abc}{8R^3} =$$

$$= \frac{4RF}{8R^3} = \frac{F}{2R^2} = \frac{rs}{2R^2}$$

Remains to prove that:

$$\frac{9r}{2R}\sqrt{\frac{r}{6R}} \leq \frac{rs}{2R^2} \leq \frac{r(4R+r)}{2R^2\sqrt{3}}$$

$$9rR\sqrt{\frac{r}{6R}} \leq rs \leq \frac{(4R+r)r}{\sqrt{3}}$$

$$9R\sqrt{\frac{r}{6R}} \leq s \leq \frac{4R+r}{\sqrt{3}}$$

$$s \leq \frac{4R+r}{\sqrt{3}} \Leftrightarrow s\sqrt{3} \leq 4R+r \text{ (Doucet's inequality)}$$

$$9R\sqrt{\frac{r}{6R}} \leq s \Leftrightarrow s^2 \geq 81R^2 \cdot \frac{r}{6R}$$

$$\Leftrightarrow s^2 \geq \frac{27Rr}{2} \text{ (to prove)}$$

By Gerretsen's inequality:

$$s^2 \geq 16Rr - 5r^2 \geq \frac{27Rr}{2} \Leftrightarrow 32Rr - 10r^2 \geq 27Rr$$

$$\Leftrightarrow 5Rr \geq 10r^2 \Leftrightarrow R \geq 2r \text{ (Euler)}$$

Equality holds for $a = b = c$. □

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