PP44944

MIHÁLY BENCZE - ROMANIA

In all triangles ABC holds:

$$1.\ \cos\frac{A-B}{2}\cos\frac{B-C}{2}\cos\frac{C-A}{2}=\frac{s^2+r^2+2Rr}{8R^2}$$

$$2.\ \frac{r(9R-2r)}{4R^2}\le\prod_{cyc}\cos\frac{A-B}{2}\le\frac{2R^2+3Rr+2r^2}{4R^2}$$

Solution by Daniel Sitaru, Claudia Nănuţi.

$$1.\ \cos\frac{A-B}{2}=\cos\frac{A}{2}\cos\frac{B}{2}+\sin\frac{A}{2}\sin\frac{B}{2}=$$

$$=\sqrt{\frac{s(s-a)}{bc}}\cdot\sqrt{\frac{s(s-b)}{ac}}+\sqrt{\frac{(s-b)(s-c)}{bc}}\cdot\sqrt{\frac{(s-a)(s-c)}{ac}}=$$

$$=\frac{s}{c}\sqrt{\frac{(s-a)(s-b)}{ab}}+\frac{s-c}{c}\sqrt{\frac{(s-a)(s-b)}{ab}}=$$

$$=\frac{2s-c}{c}\sqrt{\frac{(s-a)(s-b)}{ab}}=\frac{a+b}{c}\sin\frac{c}{2}$$

$$\prod_{cyc}\cos\frac{A-B}{2}=\prod_{cyc}\frac{a+b}{c}\sin\frac{c}{2}=$$

$$=\frac{(a+b)(b+c)(c+a)}{abc}\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=$$

$$=\frac{2s(s^2+r^2+2Rr)}{4Rrs}\cdot\frac{r}{4R}=\frac{s^2+r^2+2Rr}{8R^2}$$

$$2.\ \frac{r(9R-2r)}{4R^2}\le\frac{s^2+r^2+2Rr}{8R^2}$$

$$2r(9R-2r)\le s^2+r^2+2Rr$$

$$s^2\ge -r^2-2Rr+18Rr-4r^2$$

$$s^2\ge 16Rr-5r^2\ \text{(GERRETSEN)}$$

$$\frac{s^2+r^2+2Rr}{8R^2}\le\frac{2R^2+3Rr+2r^2}{4R^2}$$

$$s^2+r^2+2Rr\le 4R^2+6Rr+4r^2$$

$$s^2\le 4R^2+4Rr+3r^2\ \text{(GERRETSEN)}$$

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