PP45141

MIHÁLY BENCZE - ROMANIA

In all triangles ABC holds:

$$1.\ \sum_{cyc} \frac{ar_a}{h_a} = \frac{2s(R-r)}{r}$$

$$2.\ \sum_{cyc} \frac{abr_ar_b}{h_ah_b} \geq \frac{(s^2+r^2+4Rr)^2}{4r(4R+r)}$$

Solution by Daniel Sitaru and Claudia Nănuţi.
Lemma.
In ΔABC the following relationship holds:

$$\sum_{cyc} a^2(s-b)(s-c) = 4rs^2(R-r)$$

Proof.

$$\sum_{cyc} a^2(s-b)(s-c) = \sum_{cyc} a^2(s^2 - s(b+c) + bc) =$$

$$= \sum_{cyc} a^2(s^2 - s(2s-a) + bc) =$$

$$= \sum_{cyc} a^2(s^2 - 2s^2 + as + bc) =$$

$$= \sum_{cyc} a^2(bc + as - s^2) =$$

$$= abc \sum_{cyc} a + s \sum_{cyc} a^3 - s^2 \sum_{cyc} a^2 =$$

$$= abc \cdot 2s + s^2 \cdot 2(s^2 - 3r^2 - 6Rr) - 2s^2(s^2 - r^2 - 4Rr) =$$

$$= 8Rrs^2 + 2s^4 - 6r^2s^2 - 12Rrs^2 - 2s^4 + 2s^2r^2 + 8Rrs^2 =$$

$$= 4Rrs^2 - 4r^2s^2 = 4rs^2(R-r)$$

□

Back to the problem:

$$1.\ \sum_{cyc} \frac{ar_a}{h_a} = \sum_{cyc} \frac{a \cdot \frac{F}{s-a}}{\frac{2F}{a}} = \frac{1}{2} \sum_{cyc} \frac{a^2}{s-a} =$$

$$= \frac{1}{2(s-a)(s-b)(s-c)} \sum_{cyc} a^2(s-b)(s-c) =$$

$$\overset{\text{Lemma}}{=} \frac{s}{2F^2} \cdot 4rs^2(R-r) = \frac{4rs^3(R-r)}{2r^2s^2} = \frac{2s(R-r)}{r}$$

$$2.\ \sum_{cyc} \frac{abr_a r_b}{h_a h_b} = \sum_{cyc} \frac{ab \cdot \frac{F}{s-a} \cdot \frac{F}{s-b}}{\frac{2F}{a} \cdot \frac{2F}{b}} =$$

$$= \frac{1}{4} \sum_{cyc} \frac{a^2 b^2}{(s-a)(s-b)} \overset{\text{BERGSTRÖM}}{\geq} \frac{1}{4} \cdot \frac{(ab+bc+ca)^2}{\sum_{cyc}(s-a)(s-b)} =$$

$$= \frac{(s^2+r^2+4Rr)^2}{4r(4R+r)}$$

Observation:

$$\sum_{cyc}(s-a)(s-b) = \sum_{cyc}(s^2 - s(a+b) + ab) =$$

$$= \sum_{cyc}(s^2 - s(2s-c) + ab) = \sum_{cyc}(s^2 - 2s^2 + sc + ab) =$$

$$= \sum_{cyc}(sc + ab - s^2) = s\sum_{cyc} c + \sum_{cyc} ab - 3s^2 =$$

$$= s \cdot 2s + s^2 + r^2 + 4Rr - 3s^2 = r^2 + 4Rr =$$

$$= r(4R+r)$$

□

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