PP45165

MIHÁLY BENCZE - ROMANIA

In all triangles ABC holds:

$$1.\ \sum_{cyc} \frac{w_a w_b}{\sin\frac{C}{2}} = \frac{32s^2Rr}{s^2+r^2+2Rr}$$

$$2.\ \sum_{cyc} \frac{\sin\frac{A}{2}}{w_a w_b} = \frac{5s^2+r^2+4Rr}{16s^2Rr}$$

Solution by Daniel Sitaru and Claudia Nănuţi.

$$1.\ \sum_{cyc} \frac{w_a w_b}{\sin\frac{C}{2}} = \sum_{cyc} \frac{\frac{2}{b+c}\sqrt{bcs(s-a)} \cdot \frac{2}{a+c}\sqrt{acs(s-b)}}{\sqrt{\frac{(s-a)(s-b)}{ab}}}$$

$$= 4\sum_{cyc} \frac{abcs}{(b+c)(c+a)} = 4abcs\sum_{cyc} \frac{a+b}{(a+b)(b+c)(c+a)} =$$

$$= \frac{4abcs}{(a+b)(b+c)(c+a)}\left(\sum_{cyc} a + \sum_{cyc} b\right) =$$

$$= \frac{4 \cdot 4Rrs \cdot s}{2s(s^2+r^2+2Rr)}(2s+2s) =$$

$$= \frac{64Rrs^3}{2s(s^2+r^2+2Rr)} = \frac{32Rrs^2}{s^2+r^2+2Rr}$$

$$2.\ \sum_{cyc} \frac{\sin\frac{A}{2}}{w_b w_c} = \sum_{cyc} \frac{\sqrt{\frac{(s-b)(s-c)}{bc}}}{\frac{2}{a+c}\sqrt{acs(s-b)} \cdot \frac{2}{a+b}\sqrt{abs(s-c)}} =$$

$$= \frac{1}{4}\sum_{cyc} \frac{(a+b)(a+c)}{abcs} =$$

$$= \frac{1}{4abcs}\sum_{cyc}(2s-c)(2s-b) =$$

$$= \frac{1}{4abcs}\sum_{cyc}(4s^2 - 2s(b+c) + bc) =$$

$$= \frac{1}{4 \cdot 4Rrs \cdot s}\sum_{cyc}(4s^2 - 2s(2s-a) + bc) =$$

$$= \frac{1}{16Rrs^2}\left(2s\sum_{cyc} a + \sum_{cyc} bc\right) =$$

$$= \frac{1}{16Rrs^2}(2s \cdot 2s + s^2 + r^2 + 4Rr) =$$

$$= \frac{5s^2 + r^2 + 4Rr}{16Rrs^2}$$

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