PP45166

MIHÁLY BENCZE - ROMANIA

In all triangles ABC holds:

$$\sum_{cyc} \frac{\sin\frac{A}{2}}{w_b w_c} \le \frac{3\sum_{cyc} a^2 + 5\sum_{cyc} ab}{4abc\sum_{cyc} a}$$

Solution by Daniel Sitaru and Claudia Nănuţi.
As we proved at the problem PP45165:

$$\sum_{cyc} \frac{\sin\frac{A}{2}}{w_a w_b} = \frac{5s^2 + r^2 + 4Rr}{16s^2 Rr}$$

Remains to prove that:

$$\frac{5s^2 + r^2 + 4Rr}{16s^2 Rr} \le \frac{3(a^2+b^2+c^2) + 5(ab+bc+ca)}{4abc(a+b+c)}$$

$$\frac{5s^2 + r^2 + 4Rr}{16s^2 Rr} \le \frac{3(2s^2 - 2r^2 - 8Rr) + 5(s^2 + r^2 + 4Rr)}{4 \cdot 4Rrs \cdot 2s}$$

$$\frac{5s^2 + r^2 + 4Rr}{16s^2 Rr} \le \frac{6s^2 - 6r^2 - 24Rr + 5s^2 + 5r^2 + 20Rr}{32Rrs^2}$$

$$2(5s^2 + r^2 + 4Rr) \le 11s^2 - r^2 - 4Rr$$

$$11s^2 - 10s^2 \ge r^2 + 4Rr + 2r^2 + 8Rr$$

$$s^2 \ge 12Rr + 3r^2$$

$$s^2 \overset{\text{GERRETSEN}}{\ge} 16Rr - 5r^2 \ge 12Rr + 3r^2 \text{ (to prove)}$$

$$16Rr - 12Rr \ge 5r^2 + 3r^2$$

$$4Rr \ge 8r^2$$

$$R \ge 2r \text{ (Euler)}$$

Equality holds for $a = b = c$.

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