PP45167

MIHÁLY BENCZE - ROMANIA

In any triangles ABC holds:

$$\sum_{cyc} \frac{w_a w_b}{\sin \frac{C}{2}} \leq 2s^2$$

Solution by Daniel Sitaru and Claudia Nănuţi.
As we proved at the problem PP45165:

$$\sum_{cyc} \frac{w_a w_b}{\sin \frac{C}{2}} = \frac{32s^2Rr}{s^2+r^2+2Rr}$$

We must prove that:

$$\frac{32s^2Rr}{s^2+r^2+2Rr} \leq 2s^2 \Leftrightarrow \frac{16Rr}{s^2+r^2+2Rr} \leq 1 \Leftrightarrow$$
$$\Leftrightarrow 16Rr \leq s^2+r^2+2Rr$$
$$s^2 \geq 14Rr - r^2$$

But by Gerretsen's inequality:

$$s^2 \geq 16Rr - 5r^2 \geq 14Rr - r^2 \text{ (to prove)}$$
$$16Rr - 14Rr \geq 5r^2 - r^2$$
$$2Rr \geq 4r^2$$
$$R \geq 2r \text{ (Euler)}$$

Equality holds for $a = b = c$.

□

Mathematics Department, National Economic College "Theodor Costescu", Drobeta Turnu - Severin, Romania
Email address: dansitaru63@yahoo.com