PP45172

MIHÁLY BENZCE - ROMANIA

In all triangles ABC holds:

$$\sum_{cyc} a(r_b + r_c) \geq \frac{4r(4R+r)^2}{s}$$

Solution by Daniel Sitaru and Claudia Nănuţi.

$$\sum_{cyc} a(r_b + r_c) = \sum_{cyc} a\Big(\frac{F}{s-b} + \frac{F}{s-c}\Big) =$$

$$= F\sum_{cyc} a\Big(\frac{1}{s-b} + \frac{1}{s-c}\Big) = F\sum_{cyc} a \cdot \frac{s-c+s-b}{(s-b)(s-c)} =$$

$$= F\sum_{cyc} a \cdot \frac{a}{(s-b)(s-c)} = \frac{F}{(s-a)(s-b)(s-c)} \cdot \sum_{cyc} a^2(s-a) =$$

$$= \frac{Fs}{s(s-a)(s-b)(s-c)}\left(s\sum_{cyc} a^2 - \sum_{cyc} a^3\right) =$$

$$= \frac{Fs}{F^2}(s \cdot 2(s^2 - r^2 - 4Rr) - 2s(s^2 - 3r^2 - 6Rr)) =$$

$$= \frac{s}{F} \cdot 2s(s^2 - r^2 - 4Rr - s^2 + 3r^2 + 6Rr) =$$

$$= \frac{2s^2}{rs}(2r^2 + 2Rr) = \frac{2s}{r} \cdot 2r(R+r) =$$

$$= 4s(R+r) = \frac{4(R+r)s^2}{s} \overset{\text{GERRETSEN}}{\geq}$$

$$\geq \frac{4(R+r)(16Rr - 5r^2)}{s} \geq \frac{4r(4R+r)^2}{s} \text{ (to prove)}$$

$$(R+r) \cdot r(16R - 5r) \geq (4R+r)^2 \cdot r$$

$$(R+r)(16R - 5r) \geq (4R+r)^2$$

$$16R^2 - 5Rr + 16Rr - 5r^2 \geq 16R^2 + r^2 + 8Rr$$

$$11Rr - 8Rr \geq 5r^2 + r^2$$

$$3Rr \geq 6r^2$$

$$R \geq 2r \text{ (Euler)}$$

Equality holds for $a = b = c$. □

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