PP45175

MIHÁLY BENCZE - ROMANIA

In all triangles ABC holds:

$$1.\ \sum_{cyc}\cos\frac{A}{2}\le\sqrt{\frac{3(4R+r)}{2R}}$$

$$2.\ \sum_{cyc}\sin\frac{A}{2}\le\sqrt{\frac{3(2R-r)}{2R}}$$

Solution by Daniel Sitaru and Claudia Nănuţi.

$$1.\ \sum_{cyc}\cos\frac{A}{2}=\sum_{cyc}1\cdot\cos\frac{A}{2}\overset{\text{CBS}}{\le}\sqrt{3\sum_{cyc}\cos^2\frac{A}{2}}=$$

$$=\sqrt{3(2+\frac{r}{2R})}=\sqrt{\frac{3(4R+r)}{2R}}$$

Observation:

$$\sum_{cyc}\cos^2\frac{A}{2}=\sum_{cyc}\frac{s(s-a)}{bc}=\frac{s}{abc}\sum_{cyc}a(s-a)=$$

$$=\frac{s}{4Rrs}\left(s\sum_{cyc}a-\sum_{cyc}a^2\right)=$$

$$=\frac{1}{4Rr}(s\cdot 2s-2s^2+2r^2+8Rr)=$$

$$=\frac{1}{4Rr}\cdot 2r(r+4R)=\frac{r+4R}{2R}=2+\frac{r}{2R}$$

$$2.\ \sum_{cyc}\sin\frac{A}{2}=\sum_{cyc}1\cdot\sin\frac{A}{2}\overset{\text{CBS}}{\le}\sqrt{3\sum_{cyc}\sin^2\frac{A}{2}}=$$

$$=\sqrt{3(1-\frac{r}{2R})}=\sqrt{\frac{3(2R-r)}{2R}}$$

Observation:

$$\sum_{cyc}\sin^2\frac{A}{2}=\sum_{cyc}\frac{(s-b)(s-c)}{bc}=\frac{1}{abc}\sum_{cyc}a(s-b)(s-c)=$$

$$=\frac{1}{abc}\sum_{cyc}a(s^2-s(b+c)+bc)=$$

$$=\frac{1}{abc}\sum_{cyc}a(s^2-2s^2+2sa+bc)=$$

$$= \frac{1}{abc}\left(-s^2\sum_{cyc} a + s\sum_{cyc} a^2 + 3abc\right) =$$

$$= \frac{1}{abc}(-s^2 \cdot 2s + 2s(s^2 - r^2 - 4Rr) + 12Rrs) =$$

$$= \frac{1}{abc}(-2sr^2 - 8Rrs + 12Rrs) =$$

$$= \frac{1}{4Rrs}(4Rrs - 2sr^2) =$$

$$= 1 - \frac{2sr^2}{4Rrs} = 1 - \frac{r}{2R} = \frac{2R - r}{2R}$$

Equality holds for $a = b = c$.

□

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