PP45231

MIHÁLY BENCZE - ROMANIA

If $a, b, c > 0$ then:

$$\sum_{cyc} \frac{ab + c^2}{(a+b)c} \geq 3$$

Solution by Daniel Sitaru, Claudia Nănuţi.

$$\sum_{cyc} \frac{ab + c^2}{(a+b)c} = \sum_{cyc} \frac{ab}{(a+b)c} + \sum_{cyc} \frac{c^2}{(a+b)c} =$$

$$= \frac{1}{abc} \sum_{cyc} \frac{a^2b^2}{a+b} + \sum_{cyc} \frac{c}{a+b} \overset{\text{NESBITT}}{\geq}$$

$$\geq \frac{1}{abc} \sum_{cyc} \frac{(ab)^2}{a+b} + \frac{3}{2} \overset{\text{BERGSTRÖM}}{\geq}$$

$$\geq \frac{1}{abc} \cdot \frac{(ab+bc+ca)^2}{2(a+b+c)} + \frac{3}{2} \geq 3 \Leftrightarrow$$

$$(ab+bc+ca)^2 \geq 3abc(a+b+c)$$

$$a^2b^2 + b^2c^2 + c^2a^2 + 2a^2bc + 2ab^2c + 2abc^2 \geq$$

$$\geq 3a^2bc + 3ab^2c + 3abc^2$$

$$a^2b^2 + b^2c^2 + c^2a^2 - a^2bc - ab^2c - abc^2 \geq 0$$

$$\frac{1}{2}((ab-bc)^2 + (bc-ca)^2 + (ac-ab)^2) \geq 0$$

Equality holds for $a = b = c$. $\square$

Mathematics Department, National Economic College "Theodor Costescu", Drobeta Turnu - Severin, Romania

Email address: dansitaru63@yahoo.com