PP45248

MIHÁLY BENCZE - ROMANIA

Denote ω the Brocard's angle in ΔABC then:

$$1.\ \frac{3(2R-r)}{s} \le \cot\omega \le \frac{2R^2+r^2}{sr}$$

$$2.\ \cot\omega \ge \frac{s}{3r}$$

Solution by Daniel Sitaru, Claudia Nănuţi.

$$1.\ \frac{3(2R-r)}{s} \le \cot\omega \le \frac{2R^2+r^2}{sr}$$

$$\frac{3(2R-r)}{s} \le \frac{a^2+b^2+c^2}{4F} \le \frac{2R^2+r^2}{sr}$$

$$\frac{3r(2R-r)}{rs} \le \frac{2(s^2-r^2-4Rr)}{4rs} \le \frac{2R^2+r^2}{sr}$$

$$\frac{6Rr-3r^2}{rs} \le \frac{s^2-r^2-4Rr}{2rs} \le \frac{2R^2+r^2}{rs}$$

$$\begin{cases} 12Rr-6r^2 \le s^2-r^2-4Rr \\ s^2-r^2-4Rr \le 4R^2+2r^2 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} s^2 \ge 16Rr-5r \text{ (GERRETSEN)} \\ s^2 \le 4R^2+4Rr+3r^2 \text{ (GERRETSEN)} \end{cases}$$

$$2.\ \cot\omega \ge \frac{s}{3r}$$

$$\frac{a^2+b^2+c^2}{4F} \ge \frac{s}{3r}$$

$$\frac{2(s^2-r^2-4Rr)}{4rs} \ge \frac{s^2}{3rs}$$

$$\frac{s^2-r^2-4Rr}{2rs} \ge \frac{s^2}{3rs}$$

$$3(s^2-r^2-4Rr) \ge 2s^2$$

$$3s^2-3r^2-12Rr \ge 2s^2$$

$$3s^2-2s^2 \ge 3r^2+12Rr$$

$$s^2 \ge 3r^2+12Rr$$

$$s^2 \overset{\text{GERRETSEN}}{\ge} 16Rr-5r^2 \ge 3r^2+12Rr$$

$$16Rr-12Rr \ge 5r^2+3r^2$$

$$4Rr \ge 8r^2$$

$$R \ge 2r \text{ (Euler)}$$

Equality holds for $a=b=c$.

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