PP45261

MIHÁLY BENCZE - ROMANIA

If $a, b > 0; n \in \mathbb{N}^*$ then:

$$\left(\frac{a}{b}+\frac{b}{a}\right)^n \geq \left(\frac{a}{b}\right)^n + \left(\frac{b}{a}\right)^n + 2(2^{n-1}-2)$$

Solution by Daniel Sitaru, Claudia Nănuţi.
Denote $\frac{a}{b} = x > 0 \Rightarrow \frac{b}{a} = \frac{1}{x}$.
We will prove by mathematical induction:

$$P(n): \left(x+\frac{1}{x}\right)^n \geq x^n + \frac{1}{x^n} + 2^n - 2$$

For $n = 1$:

$$x + \frac{1}{x} \geq x + \frac{1}{x} + 2 - 2 \text{ (True)}$$

Suppose $P(n)$ true. We must prove:

$$P(n+1): \left(x+\frac{1}{x}\right)^{n+1} \geq x^{n+1} + \frac{1}{x^{n+1}} + 2^{n+1} - 2$$

$$\left(x+\frac{1}{x}\right)^{n+1} = \left(x+\frac{1}{x}\right)\left(x+\frac{1}{x}\right)^n \overset{P(n)}{\geq}$$

$$\geq \left(x+\frac{1}{x}\right)\left(x^n + \frac{1}{x^n} + 2^n - 2\right) \geq x^{n+1} + \frac{1}{x^{n+1}} + 2^{n+1} - 2 \text{ (to prove)}$$

$$x^{n+1} + \frac{1}{x^{n-1}} + x^{n-1} + \frac{1}{x^{n+1}} + \left(x+\frac{1}{x}\right)(2^n - 2) \geq x^{n+1} + \frac{1}{x^{n+1}} + 2^{n+1} - 2$$

$$x^{n-1} + \frac{1}{x^{n-1}} + \left(x+\frac{1}{x}\right)(2^n - 2) \geq 2^{n+1} - 2 \text{ (to prove)}$$

$$x^{n-1} + \frac{1}{x^{n-1}} + \left(x+\frac{1}{x}\right)(2^n - 2) \overset{\text{AM-GM}}{\geq}$$

$$\geq 2\sqrt{x^{n-1}\cdot\frac{1}{x^{n-1}}} + 2\sqrt{x\cdot\frac{1}{x}}\cdot(2^n-2) = 2 + 2(2^n - 2) =$$

$$= 2 + 2^{n+1} - 4 = 2^{n+1} - 2; P(n) \to P(n+1)$$

□

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