PP45324

MIHÁLY BENCZE - ROMANIA

In all acute triangles ABC holds:

$$\sum_{cyc} \frac{a+b}{\sqrt{a^2+b^2-c^2}} \geq 6$$

Solution by Daniel Sitaru, Claudia Nănuţi.

$$\text{Let be } f:\Big(0,\frac{\pi}{2}\Big)\to\mathbb{R}; f(x)=\frac{1}{\sqrt{\cos x}}$$

$$f''(x)=\frac{1}{2\sqrt{\cos x}}+\frac{3\sin x}{4\sqrt{\cos^5 x}}>0\Rightarrow f \text{ convex.}$$

By Jensen's inequality:

$$f(A)+f(B)+f(C)\geq 3f\Big(\frac{A+B+C}{3}\Big)=3f\Big(\frac{\pi}{3}\Big)$$

$$\sum_{cyc}\frac{1}{\sqrt{\cos A}}\geq 3\cdot\frac{1}{\sqrt{\cos\frac{\pi}{3}}}=3\cdot\frac{1}{\sqrt{\frac{1}{2}}}=3\sqrt{2}$$

$$\sum_{cyc}\frac{1}{\sqrt{\cos A}}\geq 3\sqrt{2} \tag{1}$$

$$\sum_{cyc}\frac{a+b}{\sqrt{a^2+b^2-c^2}}=\sum_{cyc}\frac{a+b}{\sqrt{2ab\cos c}}\overset{\text{AM-GM}}{\geq}$$

$$\geq\sum_{cyc}\frac{2\sqrt{ab}}{\sqrt{2ab\cos c}}=\sqrt{2}\cdot\sum_{cyc}\frac{1}{\sqrt{\cos c}}\overset{(1)}{\geq}$$

$$\geq\sqrt{2}\cdot 3\sqrt{2}=6$$

Equality holds for $a=b=c$. □

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