PP45388

MIHÁLY BENCZE - ROMANIA

If $0 < a \le b < 1$ then:

$$2(b-a)+\int_a^b \ln\Big(\frac{2(x^2+1)}{x^2}\Big)dx \ge \ln\Big(\frac{b^2+1}{a^2+1}\Big)+2\arctan\frac{b-a}{1+ab}$$

Solution by Daniel Sitaru, Claudia Nănuţi.

$$2(b-a)+\int_a^b \ln\Big(\frac{2(x^2+1)}{x^2}\Big)dx \ge \ln\Big(\frac{b^2+1}{a^2+1}\Big)+2\arctan\Big(\frac{b-a}{1+ab}\Big)$$

$$2\int_a^b + \int_a^b \ln\Big(\frac{2(x^2+1)}{x^2}\Big)dx \ge \int_a^b \frac{2x}{x^2+1}dx + 2\int_a^b \frac{1}{1+x^2}dx$$

$$\int_a^b \Big(2+\ln\Big(\frac{2(x^2+1)}{x^2}\Big)\Big)dx \ge \int_a^b \frac{2x+2}{1+x^2}dx$$

$$2+\ln\Big(\frac{2(x^2+1)}{x^2}\Big) \ge \frac{2x+2}{1+x^2}$$

$$2+\ln 2+\ln(x^2+1)-2\ln x \ge \frac{2x+2}{1+x^2}$$

$$\ln(x^2+1)-2\ln x+2-\frac{2x+2}{1+x^2} \ge -\ln 2$$

$$\ln(x^2+1)-2\ln x+\frac{2+2x^2-2x-2}{1+x^2} \ge -\ln 2$$

$$\ln(x^2+1)-2\ln x+\frac{2x^2-2x}{1+x^2} \ge -\ln 2 \tag{1}$$

$$\text{Let be } f:(0,1]\to\mathbb{R}$$

$$f(x)=\ln(x^2+1)-2\ln x+\frac{2x^2-2x}{1+x^2}$$

$$f'(x)=\frac{2(x+1)^2(x-1)}{x(1+x^2)^2}\le 0$$

$$f \text{ decreasing } \Rightarrow \min_{x\in(0,1]} f(x)=f(1)=\ln 2-2\ln 2=-\ln 2$$

$$\Rightarrow f(x)\ge -\ln 2 \Rightarrow (1)$$

Equality holds for $a=b$. □

Mathematics Department, National Economic College "Theodor Costescu", Drobeta Turnu - Severin, Romania

Email address: dansitaru63@yahoo.com