PP45450

MIHÁLY BENCZE - ROMANIA

In all triangles ABC holds:

$$\sum_{cyc} a(r_b + r_c) \geq \frac{4s^3}{4R + r}$$

Solution bu Daniel Sitaru and Claudia Nănuţi.
Lemma: In all triangles ABC holds:

$$\sum_{cyc} (s-b)(s-c) = r(4R + r)$$

Proof.

$$\sum_{cyc} (s-b)(s-c) = \sum_{cyc} (s^2 - s(b+c) + bc) =$$
$$= 3s^2 - s\sum_{cyc}(2s - a) + \sum_{cyc} bc =$$
$$= 3s^2 - s \cdot 6s + s\sum_{cyc} a + s^2 + r^2 + 4Rr =$$
$$= -3s^2 + s \cdot 2s + s^2 + r(4R + r) = r(4R + r)$$

□

Back to the problem:

$$\sum_{cyc} a(r_b + r_c) = \sum_{cyc} a\left(\frac{F}{s-b} + \frac{F}{s-c}\right) = F\sum_{cyc} a\left(\frac{s-b+s-c}{(s-b)(s-c)}\right) =$$
$$= F\sum_{cyc} \frac{a^2}{(s-b)(s-c)} \overset{\text{BERGSTRÖM}}{\geq} F \cdot \frac{(a+b+c)^2}{\sum_{cyc}(s-b)(s-c)} =$$
$$\overset{\text{Lemma}}{=} \frac{F \cdot 4s^2}{r(4R + r)} = \frac{rs \cdot 4s^2}{r(4R + r)} = \frac{4s^3}{4R + r}$$

Equality holds for $a = b = c$.

□

Mathematics Department, National Economic College "Theodor Costescu", Drobeta Turnu - Severin, Romania
Email address: dansitaru63@yahoo.com