PP45604

MIHÁLY BENCZE - ROMANIA

Let $a_1, a_2, \ldots, a_n, \ldots$ be an arithmetical progression and denote $S(n) = a_1 + a_2 + \ldots + a_n$. Prove that:

$$\sum_{m=1}^{\infty} \sum_{k=2}^{n} \frac{1}{S(km) - kS(m)} = \frac{\pi^2(n-1)}{3rn}$$

where r is the ratio of the progression.

Solution by Daniel Sitaru, Claudia Nănuţi.

$$\frac{1}{S(km) - kS(m)} = \frac{1}{\frac{(a_1+a_{km})km}{2} - \frac{(a_1+a_m)mk}{2}} =$$

$$= \frac{2}{km} \cdot \frac{1}{a_1 + a_{km} - a_1 - a_m} =$$

$$= \frac{2}{km(a_1 + (km-1)r - a_1 - (m-1)r)} =$$

$$= \frac{2}{kmr(km - 1 - m + 1)} = \frac{2}{km^2 r(k-1)}$$

$$\sum_{k=2}^{n} \frac{1}{S(km) - kS(m)} = \frac{2}{m^2 r} \sum_{k=2}^{n} \frac{1}{k(k-1)} =$$

$$= \frac{2}{m^2 r} \left(\sum_{k=2}^{n} \frac{1}{k-1} - \sum_{k=2}^{n} \frac{1}{k} \right) =$$

$$= \frac{2}{m^2 r} \left(1 - \frac{1}{n} \right) = \frac{2(n-1)}{nrm^2}$$

$$\sum_{m=1}^{\infty} \sum_{k=2}^{n} \frac{1}{S(km) - kS(m)} =$$

$$= \sum_{m=1}^{\infty} \frac{2(n-1)}{nrm^2} = \frac{2(n-1)}{nr} \sum_{m=1}^{\infty} \frac{1}{m^2} =$$

$$= \frac{2(n-1)}{nr} \cdot \frac{\pi^2}{6} = \frac{\pi^2(n-1)}{3nr}$$

□

Mathematics Department, National Economic College "Theodor Costescu", Drobeta Turnu - Severin, Romania

Email address: dansitaru63@yahoo.com