PP45636

MIHÁLY BENCZE - ROMANIA

If $a, b > 0; ab = 1$ then:

$$81(1+a^2)(1+b^2) \leq 4(a+b+1)^4$$

Solution by Daniel Sitaru, Claudia Nănuţi.

$$81(1+a^2)(1+b^2) \leq 4(a+b+1)^4$$
$$81(1+a^2)\left(1+\frac{1}{a^2}\right) \leq 4\left(a+\frac{1}{a}+1\right)^4$$
$$81\frac{(1+a^2)\cdot(1+a^2)}{a^2} \leq 4\cdot\frac{(a^2+a+1)^4}{a^4}$$
$$81a^2(1+a^2)^2 \leq 4(a^2+a+1)^4$$
$$9a(1+a^2) \leq 2(a^2+a+1)^2$$
$$9a+9a^3 \leq 2a^4+2a^2+2+4a^3+4a^2+4a$$
$$2a^4-5a^3+6a^2-5a+2 \geq 0$$
$$2a^4-2a^3-3a^3+3a^2+3a^2-3a-2a+2 \geq 0$$
$$2a^3(a-1)-3a^2(a-1)+3a(a-1)-2(a-1) \geq 0$$
$$(a-1)(2a^3-3a^2+3a-2) \geq 0$$
$$(a-1)[(2a^3-2a^2)-a^2+a+(2a-2)] \geq 0$$
$$(a-1)[2a^2(a-1)-a(a-1)+2(a-1)] \geq 0$$
$$(a-1)^2(2a^2-a+2) \geq 0$$
$$(a-1)^2\left[2\left(a-\frac{1}{4}\right)^2+\frac{15}{8}\right] \geq 0$$

Equality holds for $a = b = 1$.

□

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