PP45644

MIHÁLY BENCZE - ROMANIA

If $a, b > 0; ab = 1; n \geq 2; n \in \mathbb{N}$ then:

$$n(a+b) + \frac{6}{a+b} \geq 2n+3$$

Solution by Daniel Sitaru, Claudia Nănuţi.

$$ab = 1 \Rightarrow b = \frac{1}{a}$$

$$n\Big(a + \frac{1}{a}\Big) + \frac{6}{a + \frac{1}{a}} \geq 2n+3$$

$$P(n) : \frac{n(a^2+1)}{a} + \frac{6a}{a^2+1} \geq 2n+3$$

For $n = 2$:

$$\frac{2(a^2+1)}{a} + \frac{6a}{a^2+1} \geq 2 \cdot 2 + 3$$

$$2(a^2+1)^2 + 6a^2 \geq 7a(a^2+1)$$

$$2a^4 + 4a^2 + 2 + 6a^2 - 7a^3 - 7a \geq 0$$

$$2a^4 - 7a^3 + 10a^2 - 7a + 2 \geq 0$$

$$2a^4 - 2a^3 - 5a^3 + 5a^2 + 5a^2 - 5a - 2a + 2 \geq 0$$

$$2a^3(a-1) - 5a^2(a-1) + 5a(a-1) - 2(a-1) \geq 0$$

$$(a-1)(2a^3 - 5a^2 + 5a - 2) \geq 0$$

$$(a-1)(2a^3 - 2a^2 - 3a^2 + 3a + 2a - 2) \geq 0$$

$$(a-1)^2(2a^2 - 3a + 2) \geq 0 \text{ (True)}$$

Suppose $P(n)$ true

$$P(n+1) : \frac{(n+1)(a^2+1)}{a} + \frac{6a}{a^2+1} \geq 2n+5 \text{ (to prove)}$$

$$\frac{(n+1)(a^2+1)}{a} + \frac{6a}{a^2+1} =$$

$$= \frac{n(a^2+1)}{a} + \frac{6a}{a^2+1} + \frac{a^2+1}{a} \overset{P(n)}{\geq}$$

$$\geq 2n+3 + \Big(a + \frac{1}{a}\Big) \overset{\text{AM-GM}}{\geq}$$

$$\geq 2n+3+2\sqrt{a \cdot \frac{1}{a}} = 2n+3+2 = 2n+5$$

$$P(n) \to P(n+1)$$

Equality holds for $a = b = 1$.

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