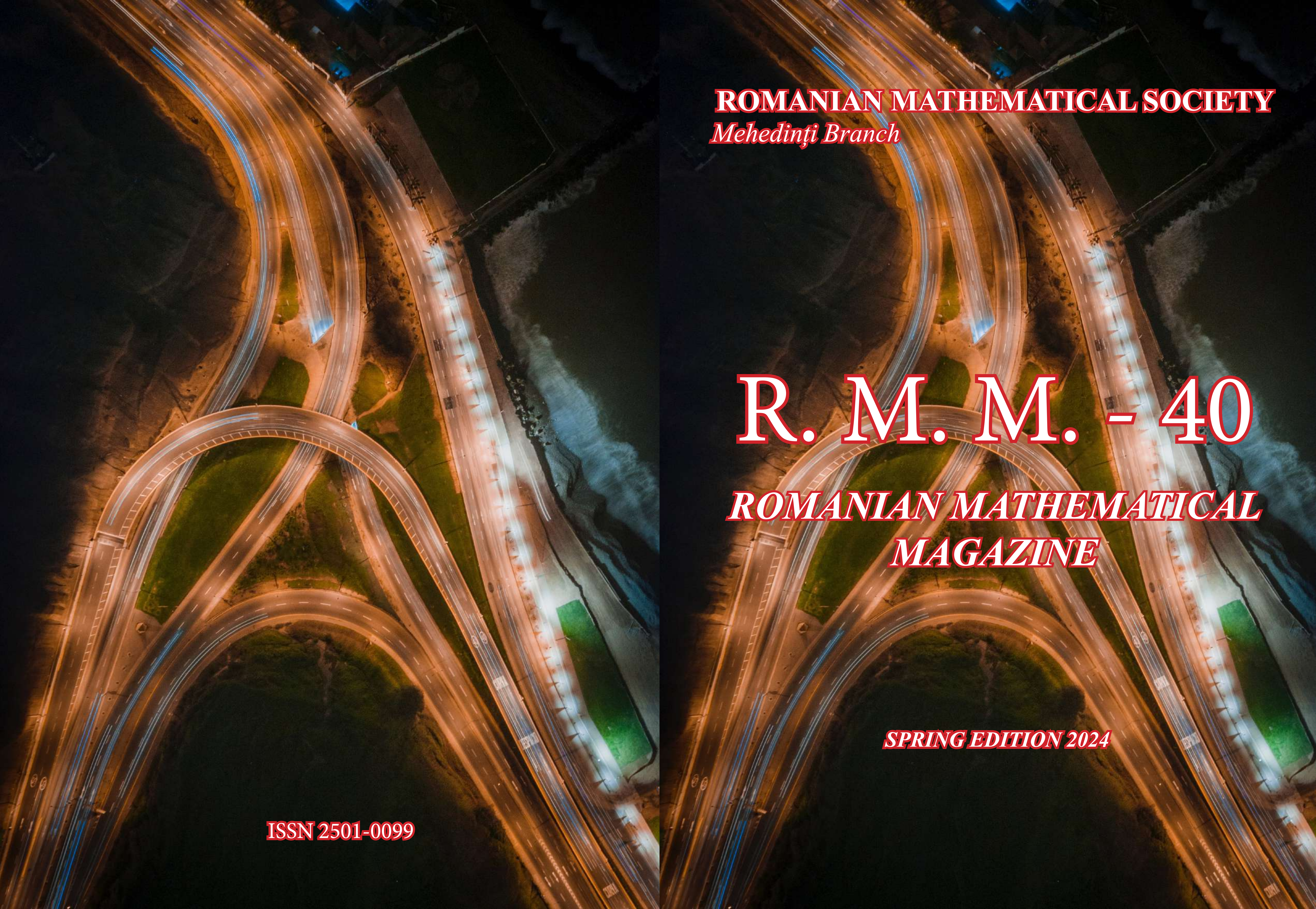

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CONTENT

ABOUT AN INEQUALITY BY MARIAN URSĂRESCU-III

By Marin Chirciu – Romania

1) In ΔABC the following relationship holds:

$$108\sum \sin^2 A\cot B\cot C\le\left[2\left(\frac{R}{r}\right)^2+1\right]^2$$

Proposed by Marian Ursărescu – Romania

Solution: *We prove the following lemma:* ***Lemma:***

2) In ΔABC the following relationship holds:

$$\sum \sin^2 A\cot B\cot C=\frac{6R^2+4Rr+r^2-s^2}{2R^2}$$

Proof. *We have* $\sum \sin^2 A\cot B\cot C=\sum(1-\cos^2 A)\cot B\cot C=$

$$=\sum \cot B\cot C-\cos^2 A\cot B\cot C=1-\prod\cos A\sum\frac{\cos A}{\sin B\sin C}=$$

$$=1-\prod\cos A\cdot\frac{1}{\prod\sin A}\sum\sin A\cos A=1-\prod\cot A\sum\frac{1}{2}\sin 2A=$$

$$=1-\frac{s^2-(2R+r)^2}{2sr}\cdot\frac{1}{2}\cdot\frac{2sr}{R^2}=1-\frac{s^2-(2R+r)^2}{2R^2}=\frac{2R^2+(2R+r)^2-s^2}{2R^2}=$$

$=\frac{6R^2+4Rr+r^2-s^2}{2R^2}$, *where above we've used the known inequalities in triangle:*

$\sum\cot B\cot C=1,\prod\cot A=\frac{s^2-(2R+r)^2}{2sr}$ *and* $\sum\sin 2A=\frac{2sr}{R^2}$.

Let's get back to the main problem.

Using the Lemma the inequality can be written:

$108\cdot\frac{6R^2+4Rr+r^2-s^2}{2R^2}\le\left[2\left(\frac{R}{r}\right)^2+1\right]^2\Leftrightarrow\frac{54(6R^2+4Rr+r^2-s^2)}{R^2}\le\frac{(2R^2+r^2)^2}{r^4}$, *which follows from Gerretsen's inequality:* $s^2\ge 16Rr-5r^2$. *It remains to prove that:*

$$\frac{54(6R^2+4Rr+r^2-16Rr+5r^2)}{R^2}\le\frac{(2R^2+r^2)^2}{r^4}\Leftrightarrow$$

$$\Leftrightarrow 4R^6+4R^4r^2-323R^2r^4+648R^2r^4-324r^6\ge 0\Leftrightarrow$$

$\Leftrightarrow(R-2r)(4R^5+8R^4r+20R^3r^2+40R^2r^3-243Rr^4+162r^5)\ge 0$, *obviously from Euler's inequality* $R\ge 2r$. *Equality holds if and only if the triangle is equilateral.*

Remark: *Inequality 1) can be strengthened:*

3) In ΔABC the following relationship holds:

$$\sum \sin^2 A \cot B \cot C \le 3\left(1-\frac{r}{R}\right)^2$$

Proposed by Marin Chirciu – Romania

Solution: *Using the Lemma and Gerretsen's inequality $s^2 \ge 16Rr - 5r^2$, we obtain:*

$$\frac{6R^2+4Rr+r^2-s^2}{2R^2} \le \frac{6R^2+4Rr+r^2-(16Rr-5r^2)}{2R^2} = \frac{6R^2-12Rr+6r^2}{2R^2} =$$

$= \frac{6(R-r)^2}{2R^2} = 3\left(1-\frac{r}{R}\right)^2$. *Equality holds if and only if the triangle is equilateral.*

Remark: *Inequality 3) is stronger than 1).*

4) In ΔABC the following relationship holds:

$$\sum \sin^2 A \cot B \cot C \le 3\left(1-\frac{r}{R}\right)^2 \le \frac{1}{108}\left[2\left(\frac{R}{r}\right)^2+1\right]^2$$

Solution: *See inequality 3) and* $3\left(1-\frac{r}{R}\right)^2 \le \frac{1}{108}\left[2\left(\frac{R}{r}\right)^2+1\right]^2 \Leftrightarrow$

$$324r^4(R-r)^2 \le R^2(2R^2+r^2)^2$$

$$\Leftrightarrow 18r^2(R-r) \le R(2R^2+r^2) \Leftrightarrow 2R^3-17Rr^2+18r^3 \ge 0 \Leftrightarrow$$

$\Leftrightarrow (R-2r)(2R^2+4Rr-9r^2) \ge 0$, *obviously from Euler's inequality $R \ge 2r$.*

Equality holds if and only if the triangle is equilateral.

Remark: *Let's find an inequality having an opposite sense:*

5) In ΔABC the following inequality holds:

$$\sum \sin^2 A \cot B \cot C \ge 1-\left(\frac{r}{R}\right)^2$$

Proposed by Marin Chirciu – Romania

Solution: *Using the Lemma and Gerretsen's inequality: $s^2 \le 4R^2+4Rr+3r^2$ we obtain:*

$$\frac{6R^2+4Rr+r^2-s^2}{2R^2} \ge \frac{6R^2+4Rr+r^2-(4R^2+4Rr+3r^2)}{2R^2} = \frac{2R^2-2r^2}{2R^2} = 1-\left(\frac{r}{R}\right)^2$$

Equality holds if and only if the triangle is equilateral.

Remark: *We can write the double inequality:*

6) In ΔABC the following relationship holds:

$$1-\left(\frac{r}{R}\right)^2 \le \sum \sin^2 A \cot B \cot C \le 3\left(1-\frac{r}{R}\right)^2$$

Proposed by Marin Chirciu – Romania

Solution:*See inequalities 3) and 5). Equality holds if and only if the triangle is equilateral.*

Reference: Romanian Mathematical Magazine-www.ssmrmh.ro

ABOUT AN INEQUALITY BY FLORICĂ ANASTASE-III

By Marin Chirciu-Romania

1) In acute ΔABC the following relationship holds:

$$\sum_{cyc} \frac{a}{b^2+c^2-a^2} > \frac{3R}{2F}$$

Proposed by Florică Anastase-Romania

Solution. Lemma. **2) In acute ΔABC the following relationship holds:**

$$\sum_{cyc} \frac{a}{b^2+c^2-a^2} = \frac{8R^2+6Rr^2-s^2}{2s[s^2-(2R+r)^2]}$$

Proof. We have:

$$\sum_{cyc} \frac{a}{b^2+c^2-a^2} = \frac{\sum a(a^2+b^2-c^2)(a^2+c^2-b^2)}{\prod(b^2+c^2-a^2)} = \frac{16sr^2(8R^2+6Rr+r^2-s^2)}{32s^2r^2[s^2-(2R+r)^2]} =$$

$$= \frac{8R^2+6Rr+r^2-s^2}{2s[s^2-(2R+r)^2)]}$$

Which follows from $\sum a(a^2+b^2-c^2)(a^2+c^2-b^2) = 16sr^2(8R^2+6Rr+r^2-s^2)$ and

$$\prod(b^2+c^2-a^2) = 32s^2r^2[s^2-(2R+r)^2]$$

Let's get back to the main problem. Using Lemma, inequality becomes as:

$\frac{8R^2+6Rr+r^2-s^2}{2s[s^2-(2R+r)^2)]} \ge \frac{3R}{2F} \Leftrightarrow 12R^3+12R^2r+9Rr^2+r^3 \ge s^2(3R+r)$, which follows from Gerretsen inequality $s^2 \le 4R^2+4Rr+3r^2$. Remains to prove that:

$$12R^3+12R^2r+9Rr^2+r^3 \ge (4R^2+4Rr+3r^2)(3R+r) \Leftrightarrow 2R^2-2Rr-r^2 \ge 0,$$

which is obviously true from $R \ge 2r$ $(Euler)$.

3) In acute ΔABC the following relationship holds:

$$\sum_{cyc}\frac{a}{b^2+c^2-a^2}\ge\frac{9R}{4F}$$

Marin Chirciu

Solution. Lemma. **4) In acute ΔABC the following relationship holds:**

$$\sum_{cyc}\frac{a}{b^2+c^2-a^2}=\frac{8R^2+6Rr^2-s^2}{2s[s^2-(2R+r)^2]}$$

We have:

$$\sum_{cyc}\frac{a}{b^2+c^2-a^2}=\frac{\sum a(a^2+b^2-c^2)(a^2+c^2-b^2)}{\prod(b^2+c^2-a^2)}=\frac{16sr^2(8R^2+6Rr+r^2-s^2)}{32s^2r^2[s^2-(2R+r)^2]}=$$

$$=\frac{8R^2+6Rr+r^2-s^2}{2s[s^2-(2R+r)^2)]}$$

Which follows from $\sum a(a^2+b^2-c^2)(a^2+c^2-b^2)=16sr^2(8R^2+6Rr+r^2-s^2)$ and

$$\prod(b^2+c^2-a^2)=32s^2r^2[s^2-(2R+r)^2]$$

Let's get back to the main problem. Using Lemma, inequality becomes as:

$$\frac{8R^2+6Rr+r^2-s^2}{2s[s^2-(2R+r)^2)]}\ge\frac{9R}{4F}\Leftrightarrow 36R^3+52R^2r+21Rr^2+2r^3\ge s^2(9R+2r)$$

Which follows from $s^2\le 4R^2+4Rr+3r^2$ $(Gerretsen)$. Remains to prove that:

$$36R^3+52R^2r+21Rr^2+2r^3\ge(4R^2+4Rr+3r^2)(9R+2r)\Leftrightarrow$$

$4R^2-7Rr-2r^2\ge 0\Leftrightarrow(R-2r)(4R+r)\ge 0$, which is obviously true from

$R\ge 2r$ $(Euler)$. Equality holds if and only if triangle is equilateral.

Remark. The problem can be much stronger.

5) In acute ΔABC the following relationship holds:

$$\sum_{cyc}\frac{a}{b^2+c^2-a^2}=\frac{1}{2s}\left(\frac{2R^2}{r^2}+\frac{R}{r}-1\right)$$

Marin Chirciu

Solution. Lemma. **6) In acute ΔABC the following relationship holds:**

$$\sum_{cyc}\frac{\boldsymbol{a}}{\boldsymbol{b^2+c^2-a^2}}=\frac{\boldsymbol{8R^2+6Rr^2-s^2}}{\boldsymbol{2s[s^2-(2R+r)^2]}}$$

We have:

$$\sum_{cyc}\frac{a}{b^2+c^2-a^2}=\frac{\sum a(a^2+b^2-c^2)(a^2+c^2-b^2)}{\prod(b^2+c^2-a^2)}=\frac{16sr^2(8R^2+6Rr+r^2-s^2)}{32s^2r^2[s^2-(2R+r)^2]}=$$

$$=\frac{8R^2+6Rr+r^2-s^2}{2s[s^2-(2R+r)^2)]}$$

Which follows from $\sum a(a^2+b^2-c^2)(a^2+c^2-b^2)=16sr^2(8R^2+6Rr+r^2-s^2)$ and

$$\prod(b^2+c^2-a^2)=32s^2r^2[s^2-(2R+r)^2]$$

Let's get back to the main problem. Using Lemma, inequality becomes as:

$$\sum_{cyc}\frac{a}{b^2+c^2-a^2}=\frac{8R^2+6Rr^2-s^2}{2s[s^2-(2R+r)^2]}\overset{Gerretsen}{\geq}$$

$$\geq\frac{8R^2+6Rr+r^2-4R^2-4Rr-3r^2}{2s[4R^2+4Rr+3r^2-4R^2-4Rr-3r^2]}=\frac{4R^2+2Rr-2r^2}{2s(2r^2)}=\frac{1}{2s}\left(\frac{2R^2}{r^2}+\frac{R}{r}-1\right)\Leftrightarrow$$

$36R^3+52R^2r+21Rr^2+2r^3\geq s^2(9R+2r)$, which follows from

$s^2\leq 4R^2+4Rr+3r^2(Gerretsen)$. Remains to prove that:

$$36R^3+52R^2r+21Rr^2+2r^3\geq(4R^2+4Rr+3r^2)(9R+2r)\Leftrightarrow$$

$4R^2-7Rr-2r^2\geq 0\Leftrightarrow(R-2r)(4R+r)\geq 0$, which is true from $R\geq 2r$ $(Euler)$.

Equality holds if and only if triangle is equilateral.

Remark. We can write that:

7) In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\boldsymbol{a}}{\boldsymbol{b^2+c^2-a^2}}=\frac{\boldsymbol{1}}{\boldsymbol{2s}}\left(\frac{\boldsymbol{2R^2}}{\boldsymbol{r^2}}+\frac{\boldsymbol{R}}{\boldsymbol{r}}-\boldsymbol{1}\right)\geq\frac{\boldsymbol{9R}}{\boldsymbol{4F}}$$

Marin Chirciu

Solution. See inequality 5) and $\frac{1}{2s}\left(\frac{2R^2}{r^2}+\frac{R}{r}-1\right)\geq\frac{9R}{4F}\Leftrightarrow 4R^2-7Rr-2r^2\geq 0\Leftrightarrow$

$(R-2r)(4R+r)\geq 0$, which is obviously true from $R\geq 2r$ $(Euler)$. Equality holds if and only if triangle is equilateral.

Reference: ROMANIAN MATHEMATICAL MAGAZINE-Interactive Journal, www.ssmrmh.ro

INTEGRALS INVOLVING $\zeta(2)$ AND $\zeta(4)$

By Said Attaoui-Algerie

Prove that:

$$\boldsymbol{J=\int_0^\infty\int_0^\infty\frac{(xt)^2e^{-(x+\sqrt{2}t)}}{\left(1+e^{-x}+e^{-\sqrt{2}t}+e^{-(x+\sqrt{2}t)}\right)^2}dx\,dt=\frac{5}{\sqrt{2}}\zeta(4)}$$

Solution. Observe that:

$$1+e^{-x}+e^{-\sqrt{2}t}+e^{-(x+\sqrt{2}t)}=1+e^{-x}+e^{-\sqrt{2}t}(1+e^{-x})=(1+e^{-x})\left(1+e^{-\sqrt{2}t}\right)$$

$$\text{So,}J=\int_0^\infty\int_0^\infty\frac{(xt)^2e^{-(x+\sqrt{2}t)}}{\left[(1+e^{-x})\left(1+e^{-\sqrt{2}t}\right)\right]^2}dx\,dt=\int_0^\infty\int_0^\infty\frac{x^2t^2e^{-(x\sqrt{2}t)}}{(1+e^{-x})^2\left(1+e^{-\sqrt{2}t}\right)^2}dx\,dt=$$

$$=\left(\int_0^\infty\frac{x^2e^{-x}}{(1+e^{-x})^2}dx\right)\left(\int_0^\infty\frac{t^2e^{-\sqrt{2}t}}{\left(1+e^{-\sqrt{2}t}\right)^2}dt\right)$$

$$\text{Now, for } a>0,\text{let:}J_a=\int_0^\infty\frac{x^2e^{-ax}}{(1+e^{-ax})^2}dx\text{ and }I=\int_0^\infty\frac{xe^{-x}}{1+e^{-x}}dx.$$

Since $e^{-x}<1$, we have by applying the geometric series:

$$I=\int_0^\infty\frac{xe^{-x}}{1+e^{-x}}dx=\int_0^\infty xe^{-x}\left(\sum_{n=0}^\infty(-1)^ne^{-nx}\right)dx=$$

$$=\sum_{n=0}^\infty(-1)^n\left(\int_0^\infty xe^{-(n+1)x}\,dx\right)=\sum_{n=0}^\infty\frac{(-1)^n}{(n+1)^2}$$

Since for all $b>0; c>0$ we have: $\int_0^\infty x^ce^{-bx}\,dx=\frac{\Gamma(c+1)}{b^{c+1}}$. Therefore,

$$I=\sum_{n=0}^\infty\frac{(-1)^n}{(n+1)^2}=\sum_{n=1}^\infty\frac{(-1)^{n-1}}{n^2}=-\sum_{n=0}^\infty\frac{1}{(2n)^2}+\sum_{n=0}^\infty\frac{1}{(2n+1)^2}=$$

$$=-\sum_{n=0}^\infty\frac{1}{(2n)^2}+\zeta(2)-\sum_{n=0}^\infty\frac{1}{(2n)^2}=\frac{1}{2}\zeta(2)$$

Now, making change of the variable $x\to ax, a>0$ in I, we get:

$$I_a=\int_0^\infty\frac{xe^{-ax}}{1+e^{-ax}}dx=\frac{1}{2a^2}\zeta(2)$$

Differentiate w.r.t. a, we obtain:

$$\frac{\partial}{\partial a}I_a=\frac{\partial}{\partial a}\left(\int_0^\infty\frac{xe^{-ax}}{1+e^{-ax}}dx\right)=\int_0^\infty\frac{x(-xe^{-ax}(1+e^{-ax})+xe^{-2ax})}{(1+e^{-ax})^2}dx=$$
$$=-\int_0^\infty\frac{x^2e^{-ax}}{(1+e^{-ax})^2}dx.$$

$$\text{Therefore, }J_a=\frac{1}{a^3}\zeta(2).\text{ Thus, }J=J_1\cdot J_{\sqrt{2}}=\frac{1}{2\sqrt{2}}\zeta^2(2)$$

$$\text{Recall that: }\zeta^2(2)=\left(\frac{\pi^2}{6}\right)^2=\frac{\pi^4}{36}.\text{ Finally, we get: }J=\frac{5}{\sqrt{2}}\zeta(4).$$

Remark. We have:

$$\int_0^1\int_0^1\frac{\log^2 u\log^2 v}{(1+u+v+uv)^2}du\,dv=\zeta^2(2)$$

***Proof of Remark*.** Making the double changes of variables $u=e^{-x}$ and $v=e^{-\sqrt{2}t}$ in J, we get:

$$J=\int_0^\infty\int_0^\infty\frac{(xt)^2e^{-(x+\sqrt{2}t)}}{\left(1+e^{-x}+e^{-\sqrt{2}t}+e^{-(x+\sqrt{2}t)}\right)^2}dx\,dt=$$

$$=\int_0^1\int_0^1\frac{2uv\log^2 u\log^2 v}{(1+u+v+uv)^2}\left(\frac{du}{u}\frac{dv}{\sqrt{2}v}\right)=\sqrt{2}\int_0^1\int_0^1\frac{\log^2 u\log^2 v}{(1+u+v+uv)^2}du\,dv$$

$$\text{Hence, }\int_0^1\int_0^1\frac{\log^2 u\log^2 v}{(1+u+v+uv)^2}du\,dv=\frac{5}{4}\zeta(4)=\zeta^2(2).$$

$$\text{This leads, }\zeta^2(2)=\int_0^1\int_0^1\frac{\log^2 u\log^2 v}{\left(1+u+v(1+u)\right)^2}du\,dv=$$

$$=\int_0^1\int_0^1\frac{\log^2 u\log^2 v}{[(1+u)(1+v)]^2}du\,dv=\int_0^1\int_0^1\left(\frac{\log u}{1+u}\cdot\frac{\log v}{1+v}\right)^2du\,dv=\left(\int_0^1\left(\frac{\log u}{1+u}\right)^2du\right)^2=\zeta(2)$$

Reference: ROMANIAN MATHEMATICAL MAGAZINE-Interactive Journal, www.ssmrmh.ro

NEW INEQUALITIES IN TRIANGLE

By Tran Quoc Anh-Vietnam

1. Topic: In acute ΔABC the following relationship holds:

$$\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\le\frac{R+r}{r}$$

Equality holds for ΔABC equilateral.

2. Lemma: In ΔABC the following relationship holds:

$$am_a^2+bm_b^2+cm_c^2=\frac{s(s^2+2Rr+5r^2)}{2}$$

Proof of Lemma: We have:

$$am_a^2+bm_b^2+cm_c^2=a\left(\frac{2(b^2+c^2)-a^2}{4}\right)+b\left(\frac{2(c^2+a^2)-b^2}{4}\right)+c\left(\frac{2(a^2+b^2)-c^2}{4}\right)$$

$$=a\left(\frac{b^2+c^2+2bc\cos A}{4}\right)+b\left(\frac{c^2+a^2+2ca\cos B}{4}\right)+c\left(\frac{a^2+b^2+2ab\cos C}{4}\right)$$

$$=\frac{(a+b+c)(ab+bc+ca)}{4}-\frac{3abc}{4}+\frac{abc(\cos A+\cos B+\cos C)}{2}$$

$$=\frac{2s(s^2+r^2+4Rr)}{4}-\frac{3}{4}\cdot 4Rrs+\frac{4Rrs}{2}\left(\frac{R+r}{r}\right)=\frac{s(s^2+2Rr+5r^2)}{2}$$

Proof. We have: $h_a=\frac{2F}{a}, h_b=\frac{2F}{b}, h_c=\frac{2F}{c}$. Applying Cauchy's inequality:

$$\left(\frac{am_a+bm_b+cm_c}{2F}\right)^2\le(a+b+c)\left(\frac{am_a^2+bm_b^2+cm_c^2}{4F^2}\right)=$$

$$=2s\left(\frac{s(s^2+2Rr+5r^2)}{2\cdot 4F^2}\right)=\frac{s^2(s^2+2Rr+5r^2)}{4F^2}$$

We will prove that:

$$\frac{s^2(s^2+2Rr+5r^2)}{4F^2}\le\left(\frac{R+r}{r}\right)^2;(1)$$

In fact: $(1)\Leftrightarrow s^2+2Rr+5r^2\le 4R^2+8Rr+4r^2\Leftrightarrow s^2\le 4R^2+6Rr-r^2$

Alternatively: $s^2\le 4R^2+4Rr+3r^3\ (Gerretsen)$. We will prove that: $4R^2+4Rr+3r^2\le 4R62+6Rr-r^2;(2)$

In fact: $(2)\Leftrightarrow 4r^2\le 2Rr\Leftrightarrow 2r\le R\ (Euler);(3)$

Therefore: $\left(\frac{am_a+bm_b+cm_c}{2F}\right)^2\le\left(\frac{R+r}{r}\right)^2\Leftrightarrow\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}=\frac{am_a+bm_b+cm_c}{2F}\le\frac{R+r}{r}=1+\frac{R}{r}$

Identity occurs if and only if $\frac{\sqrt{a}m_a}{\sqrt{a}}=\frac{\sqrt{b}m_b}{\sqrt{b}}=\frac{\sqrt{c}m_c}{\sqrt{c}}, R=2r$ and $s^2=4R^2+4Rr+3r^2\Leftrightarrow$

ΔABC equilateral.

Reference: ROMANIAN MATHEMATICAL MAGAZINE-Interactive Journal, www.ssmrmh.ro

FEW AMAZING TRIPLE INTEGRALS

By Asmat Qatea-Afghanistan

Find:

$$\boldsymbol{\Omega = \int_0^1 \int_0^1 \int_0^1 \sqrt{x^2+y^2+z^2}\, dxdydz}$$

Solution.

$$(*):\begin{cases} dx = dv \Rightarrow x = v \\ u = \sqrt{x^2+y^2+z^2} \end{cases} \Rightarrow du = \frac{x}{\sqrt{x^2+y^2+z^2}}dx$$

$$\frac{\Omega}{3} = \int_0^1 \int_0^1 \int_0^1 \frac{x^2}{\sqrt{x^2+y^2+z^2}} dxdydz$$

$$\Omega = \int_0^1 \int_0^1 \int_0^1 \sqrt{x^2+y^2+z^2}\, dxdydz \overset{(*)}{=} \int_0^1 \int_0^1 x\sqrt{x^2+y^2+z^2}\Big|_0^1 dydz = \frac{\Omega}{3}$$

$$\frac{4}{3}\Omega = \int_0^1 \int_0^1 \sqrt{1+y^2+z^2}\, dydz =$$

$$= \int_0^1 \int_0^1 \frac{1}{\sqrt{1+y^2+z^2}} dydz + 2\int_0^1 \int_0^1 \frac{y^2}{\sqrt{1+y^2+z^2}} dydz\,; (I)$$

$$(**):\begin{cases} dy = dv \Rightarrow y = v \\ u = \sqrt{1+y^2+z^2} \end{cases} \Rightarrow du = \frac{y}{\sqrt{1+y^2+z^2}}dy$$

$$\frac{4}{3}\Omega = \int_0^1 \int_0^1 \sqrt{1+y^2+z^2}\, dydz \overset{(**)}{=} \int_0^1 y\sqrt{1+y^2+z^2}\Big|_0^1 dz - \int_0^1 \int_0^1 \frac{y^2}{\sqrt{1+y^2+z^2}} dydz$$

$$\frac{8}{3}\Omega = 2\int_0^1 \sqrt{2+z^2} dz - 2\int_0^1 \int_0^1 \frac{y^2}{\sqrt{1+y^2+z^2}} dydz\,; (II)$$

$$4\Omega = 2\int_0^1 \sqrt{2+z^2}\, dz + \underbrace{\int_0^1 \int_0^1 \frac{1}{\sqrt{1+y^2+z^2}} dydz}_{P};\quad (I)+(II)$$

$$\int \sqrt{a^2+x^2}\, dx = \frac{x}{2}\sqrt{a^2+x^2} - \frac{a^2}{2}\log\left(x+\sqrt{a^2+x^2}\right) + C$$

$$\int_0^1 \sqrt{2+z^2}\, dz = \frac{\sqrt{3}}{2} + \log\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)$$

$$4\Omega=\sqrt{3}+2\log\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)+P$$

$$P=2\log(1+\sqrt{3})-\log 2-\frac{\pi}{6}$$

$$4\Omega=\sqrt{3}+2\log(1+\sqrt{3})-\log 2+2\log(1+\sqrt{3})-\log 2-\frac{\pi}{6}$$

Therefore,

$$\Omega=\int_0^1\int_0^1\int_0^1\sqrt{x^2+y^2+z^2}\,dxdydz=\frac{\sqrt{3}}{4}+\log\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)-\frac{\pi}{24}$$

Now, let's find:

$$\mathrm{P}=\int_0^1\int_0^1\frac{1}{\sqrt{1+y^2+z^2}}dydz$$

$$\int\frac{dx}{\sqrt{a^2+x^2}}=\log\left(x+\sqrt{a^2+x^2}\right)+C$$

$$P=\int_0^1\int_0^1\frac{1}{\sqrt{1+y^2+z^2}}dydz=\int_0^1\log\left(y+\sqrt{1+y^2+z^2}\right)\Big|_0^1dydz=$$

$$=\int_0^1\log\left(1+\sqrt{2+z^2}\right)dz-\frac{1}{2}\int_0^1\log(1+z^2)\,dz$$

$$\int\log(1+x^2)\,dx=x\log(1+x^2)+2\tan^{-1}x-2x+C$$

$$P=\underbrace{\int_0^1\log\left(1+\sqrt{2+z^2}\right)dz}_{B}-\frac{1}{2}\left(\log 2+\frac{\pi}{2}-2\right)$$

$$(***):\begin{cases}dz=dv\Rightarrow z=v\\ u=\log\left(1+\sqrt{2+z^2}\right)\end{cases}\Rightarrow du=\frac{z}{2+z^2+\sqrt{2+z^2}}$$

$$B=\int_0^1\log\left(1+\sqrt{2+z^2}\right)dz=\log(1+\sqrt{3})-\underbrace{\int_0^1\frac{z^2}{2+z^2+\sqrt{2+z^2}}dz}_{C}$$

$$C=\int_0^1\frac{z^2(z^2+2)-z^2\sqrt{2+z^2}}{(z^2+2)(z^2+1)}dz=\int_0^1\frac{z^2}{z^2+1}dz-\int_0^1\frac{z^2\sqrt{2+z^2}}{z^2+1}dz+\int_0^1\frac{z^2\sqrt{2+z^2}}{z^2+2}dz$$

$$=\int_0^1\frac{z^2}{z^2+1}dz-\int_0^1\frac{z^2\sqrt{2+z^2}}{z^2+1}dz+\int_0^1\frac{z^2}{\sqrt{2+z^2}}dz=$$

$$=\underbrace{\int_0^1 \frac{z^2}{z^2+1}dz}_{N}-\int_0^1 \sqrt{2+z^2}\,dz+\underbrace{\int_0^1 \frac{\sqrt{2+z^2}}{z^2+1}dz}_{M}+\underbrace{\int_0^1 \frac{z^2}{\sqrt{2+z^2}}dz}_{S}=$$

$$=1-\frac{\pi}{4}-\frac{\sqrt{3}}{2}-\log\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)+\frac{\pi}{6}+\log\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)+\frac{\sqrt{3}}{2}-\log\left(\frac{1+\sqrt{3}}{2}\right)$$

Hence,

$$C=1-\frac{\pi}{12}-\log\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)$$

$$N=\int_0^1 \frac{x^2}{1+x^2}dx=1-\frac{\pi}{4}$$

$$S=\int_0^1 \frac{z^2}{\sqrt{2+z^2}}dz=\int_0^1 \sqrt{2+z^2}\,dz-2\int_0^1 \frac{1}{\sqrt{2+z^2}}dz=$$

$$=\frac{\sqrt{3}}{2}+\log\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)-2\log\left(z+\sqrt{2+z^2}\right)\Big|_0^1=$$

$$=\frac{\sqrt{3}}{2}+\log\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)-2\log\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)=\frac{\sqrt{3}}{2}-\log\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)$$

$$M=\int_0^1 \frac{\sqrt{2+z^2}}{z^2+1}dz\overset{z=\sqrt{2}\tan x}{=}\int_0^a \frac{\sqrt{2}\sec x}{2\tan^2 x+1}\sqrt{2}\sec^2 x\,dx=$$

$$=\int_0^a \frac{2\sec^3 x}{2\tan^2 x+1}dx=\int_0^a \frac{\sec x\left(1+(1+2\tan^2 x)\right)}{2\tan^2 x+1}dx=$$

$$=\int_0^a \frac{\sec x}{2\tan^2 x+1}dx+\int_0^a \sec x\,dx=\int_0^a \frac{\cos x}{2\sin^2 x+\cos^2 x}dx+\log(sec\,x+tan\,x)|_0^a=$$

$$=\int_0^a \frac{\cos x}{\sin^2 x+1}dx+\log(\sec a+\tan a)\overset{\sin x=a}{=}$$

$$=\int_0^{\sin a} \frac{du}{u^2+1}+\log\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)=\tan^{-1}(\sin a)+\log\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)$$

$$\sin\left(tan^{-1}\left(\frac{1}{\sqrt{2}}\right)\right)=\sqrt{1-\frac{1}{1+\frac{1}{2}}}=\sqrt{1-\frac{2}{3}}=\frac{1}{\sqrt{3}}$$

$$M=\frac{\pi}{6}+\log\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)$$

$$P = B - \frac{1}{2}\log 2 - \frac{\pi}{4} + 1 = \log(1+\sqrt{3}) - 1 + \frac{\pi}{12} + \log(1+\sqrt{3}) - \log\sqrt{2} =$$

$$= 2\log(1+\sqrt{3}) - 1 + \frac{\pi}{12} - \frac{1}{2}\log 2 = 2\log(1+\sqrt{3}) - \log 2 - \frac{\pi}{6}$$

REFERENCE: ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

A GENERALIZATION FOR A DANIEL SITARU INEQUALITY

By Marin Chirciu-Romania

If $x, y, z > 0$ and $0 \le \lambda \le 8$ then:

$$\sum_{cyc}(x^2+y^2)z + \lambda xyz\sum_{cyc}\frac{xy}{(x+y)^2} \ge \frac{3}{4}(\lambda+8)xyz$$

Proposed by Marin Chirciu-Romania

Note: For $\lambda = 1$ we get the proposed problem by *Daniel Sitaru* in R.M.M. 12/2021

If $x, y, z > 0$ then:

$$\sum_{cyc}(x^2+y^2)\,z + xyz\sum_{cyc}\frac{xy}{(x+y)^2} \ge \frac{27}{4}xyz$$

Solution:

$$\sum_{cyc}(x^2+y^2)z + \lambda xyz\sum_{cyc}\frac{xy}{(x+y)^2} \ge \frac{3}{4}(\lambda+8)xyz \,\Big|: xyz \Rightarrow$$

$$\sum_{cyc}\frac{x^2+y^2}{xy} + \lambda\sum_{cyc}\frac{xy}{(x+y)^2} \ge \frac{3}{4}(\lambda+8) \Leftrightarrow \sum_{cyc}\left(\frac{x}{y}+\frac{y}{x}\right) + \lambda\sum_{cyc}\frac{1}{\frac{x}{y}+\frac{y}{x}+2} \ge \frac{3}{4}(\lambda+8)$$

which follows from

Lemma: If $t \ge 2$ then:

$$t + \frac{\lambda}{t+2} \ge \frac{\lambda+8}{4}$$

Proof. We have:

$$t+\frac{\lambda}{t+2}\ge\frac{\lambda+8}{4}\Leftrightarrow 4t^2-\lambda t+2\lambda-16\ge 0\Leftrightarrow (t-2)(4t+8-\lambda)\ge 0$$

which is true from $t\ge 2$ and $4t+8-\lambda\ge 0$. Equality holds for $x=y=z$.

REFERENCE: ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

ABOUT AN INEQUALITY FROM A.P.M.O. 2OO4

By D.M. Bătinețu-Giurgiu, Mihaly Bencze, Florică Anastase-Romania

In A.P.M.O. 2004 was proposed problem A.P.M.O.-2004/5:

If $a, \boldsymbol{b}, \boldsymbol{c}>0$**, then the following relationship holds:**

$$(\boldsymbol{a}^2+2)(\boldsymbol{b}^2+2)(\boldsymbol{c}^2+2)\ge 3(\boldsymbol{ab}+\boldsymbol{bc}+\boldsymbol{ca});(*)$$

This problem has generalized by Arkady M. Alt in [1] in the form:

If $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{t}>0$, then the following relationship holds:

$$(\boldsymbol{x}^2+\boldsymbol{t}^2)(\boldsymbol{y}^2+\boldsymbol{t}^2)(\boldsymbol{z}^2+\boldsymbol{t}^2)\ge\frac{3}{4}\boldsymbol{t}^4(\boldsymbol{x}+\boldsymbol{y}+\boldsymbol{z})^2;(**)$$

Next, we will to prove the following problem:

If $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{t}>0$ then the following relationship holds:

$$(\boldsymbol{a}^2+\boldsymbol{x}^2\boldsymbol{t}^2)(\boldsymbol{b}^2+\boldsymbol{y}^2\boldsymbol{t}^2)(\boldsymbol{c}^2+\boldsymbol{z}^2\boldsymbol{t}^2)\ge\frac{3}{4}\boldsymbol{t}^4(\boldsymbol{ayz}+\boldsymbol{bzx}+\boldsymbol{cxy})^2;(***)$$

To prove the above inequality, first, we prove the next Lemma:

Lemma: If $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}>0$, then the following relationship holds:

$$(\boldsymbol{u}^2+\mathbf{1})(\boldsymbol{v}^2+\mathbf{1})(\boldsymbol{w}^2+\mathbf{1})\ge\frac{3}{4}(\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w})^2;\ \ (\mathbf{1})$$

Proof. We have:

$$(i):(u^2+1)(v^2+1)\ge(u+v)^2\Leftrightarrow u^2v^2+u^2+v^2+1\ge u^2+2uv+v^2$$

$$u^2v^2-2uv+1\ge 0\Leftrightarrow(uv-1)^2\ge 0\text{ true!}$$

$$(ii):(u^2+1)(v^2+1)\ge\frac{3}{4}((u+v)^2+1)\Leftrightarrow$$

$$u^2v^2+u^2+v^2+1\ge\frac{3}{4}(u^2+2uv+v^2+1)\Leftrightarrow$$

$$4u^2v^2+4u^2+4v^2+4\ge 3u^2+3v^2+6uv+3\Leftrightarrow$$

$$4u^2v^2-4uv+1+u^2-2uv+v^2\ge 0\Leftrightarrow(2uv-1)^2+(u-v)^2\ge 0 \text{ true!}$$

Hence, we have:

$$(u^2+1)(v^2+1)(w^2+1)\overset{(ii)}{\ge}\frac{3}{4}((u+v)^2+1)(w^2+1)\overset{(i)}{\ge}\frac{3}{4}\big((u+v)+w\big)^2\Leftrightarrow$$

$$(u^2+1)(v^2+1)(w^2+1)\ge\frac{3}{4}(u+v+w)^2$$

Now, let's prove the inequality $(**)$, we have:

$$(a^2+x^2t^2)(b^2+y^2t^2)(c^2+z^2t^2)=x^2y^2z^2t^2\left(\left(\frac{a}{xt}\right)^2+1\right)\left(\left(\frac{b}{yt}\right)^2+1\right)\left(\left(\frac{c}{zt}\right)^2+1\right);\ \ (2)$$

If we take in (1): $u=\frac{a}{xt},v=\frac{b}{yt},w=\frac{c}{zt}$, we get:

$$\left(\left(\frac{a}{xt}\right)^2+1\right)\left(\left(\frac{b}{yt}\right)^2+1\right)\left(\left(\frac{c}{zt}\right)^2+1\right)\ge\frac{3}{4}\left(\frac{a}{xt}+\frac{b}{yt}+\frac{c}{zt}\right)^2=$$

$$=\frac{3}{4t^2}\left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}\right)^2=\frac{3}{4t^2x^2y^2z^2}(ayz+bzx+cxy)^2;\ \ (3)$$

From (2) and (3), it follows:

$$(a^2+x^2t^2)(b^2+y^2t^2)(c^2+z^2t^2)\ge x^2y^2z^2t^6\cdot\frac{3}{4t^2x^2y^2z^2}(ayz+bzx+cxy)^2$$

If in $(***)$ we take $x=y=z$, we get:

$$(a^2+x^2t^2)(b^2+x^2t^2)(c^2+x^2t^2)\ge\frac{3}{4}t^4(ax^2+by^2+cz^2)^2=$$
$$=\frac{3}{4}t^4x^4(a+b+c)^2;\ \ (4)$$

If we take $x=1$ in (4), we obtain:

$$(a^2+t^2)(b^2+t^2)(c^2+t^2)\ge\frac{3}{4}t^4(a+b+c)^2;\ \ (A.A.)$$

i.e. inequality (4) from [1]. If we take $t=\sqrt{2}$ in $(A.A.)$, we obtain:

$$(a^2+2)(b^2+2)(c^2+2)\ge\frac{3}{4}\left(\sqrt{2}\right)^4(a+b+c)^2=$$

$$=3(a+b+c)^2\ge 3\cdot 3(ab+bc+ca)=9(ab+bc+ca)$$

REFERENCE:

[1]. **Alt M. Arkady,** ABOUT AN INEQUALITY FROM A.P.M.O. 2004-NEW SOLUTION AND GENERALIZATIONS-Octogon Mathematical Magazine, Vol.27, No.1, April 2019,pag.228-232.

ABOUT AN INEQUALITY BY GEORGE APOSTOLOPOULOS-X

By Marin Chirciu-Romania

In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{w_b + w_c}{a^2(b+c)} \le \frac{\sqrt{3}}{16} \cdot \frac{R}{r^3}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution. Lemma. In ΔABC the following relationship holds:

$$w_b + w_c \le \frac{a^2+b^2+c^2}{2\sqrt{3}}\left(\frac{1}{b}+\frac{1}{c}\right)$$

Proof. We have:

$$w_b + w_c \le m_b + m_c \overset{(1)}{\le} \frac{a^2+b^2+c^2}{2\sqrt{3}b} + \frac{a^2+b^2+c^2}{2\sqrt{3}c} = \frac{a^2+b^2+c^2}{2\sqrt{3}}\left(\frac{1}{b}+\frac{1}{c}\right)$$

where (1) it follows from:

Lemma. In ΔABC the following relationship holds:

$$a \cdot m_a \le \frac{a^2+b^2+c^2}{2\sqrt{3}}$$

Proof. Using $m_a^2 = \frac{2b^2+2c^2-a^2}{4}$, we have: $m_a \le \frac{1}{2\sqrt{3}}\sum_{cyc} a^2 \Leftrightarrow a^2 \cdot m_a^2 \le \frac{1}{12}\left(\sum_{cyc} a^2\right)^2 \Leftrightarrow$

$$a^2 \cdot \frac{2b^2+2c^2-a^2}{4} \le \frac{1}{12}\left(\sum_{cyc} a^4 + 2\sum_{cyc} b^2c^2\right) \Leftrightarrow$$

$$3a^2(2b^2+2c^2-a^2) \le a^4+b^4+c^4+2b^2c^2+2c^2a^2+2a^2b^2 \Leftrightarrow$$

$$4a^4+b^4+c^4+2b^2c^2-4a^2b^2-4a^2c^2 \ge 0 \Leftrightarrow (2a^2-b^2-c^2)^2 \ge 0$$

Equality holds for $2a^2-b^2-c^2=0 \Leftrightarrow 2a^2=b^2+c^2$.

Let's get back to the main problem. Using Lemma we get:

$$LHS=\sum_{cyc}\frac{w_b+w_c}{a^2(b+c)}\overset{Lemma}{\leq}\sum_{cyc}\frac{\frac{a^2+b^2+c^2}{2\sqrt{3}}\left(\frac{1}{b}+\frac{1}{c}\right)}{a^2(b+c)}=\frac{a^2+b^2+c^2}{2\sqrt{3}abc}\sum_{cyc}\frac{1}{a}\overset{Leuberger}{\leq}$$

$$\overset{Leuberger}{\leq}\frac{a^2+b^2+c^2}{2\sqrt{3}abc}\cdot\frac{s}{3Rr}\overset{Leibniz}{\leq}\frac{9R^2}{2\sqrt{3}\cdot 4Rrs}=\frac{\sqrt{3}}{8r^2}\overset{Euler}{\leq}\frac{\sqrt{3}}{16}\cdot\frac{R}{r^3}=RHS$$

Equality holds if and only if triangle is equilateral.**Remark:** Inequality can be developed.

In ΔABC the following relationship holds:

$$\frac{\sqrt{3}}{2R^2}\leq\sum_{cyc}\frac{w_b+w_c}{a^2(b+c)}\leq\frac{\sqrt{3}}{8r^2}$$

Proposed by Marin Chirciu-Romania

Solution: For RHS, we use: **Lemma. In ΔABC the following relationship holds:**

$$w_b+w_c\leq\frac{a^2+b^2+c^2}{2\sqrt{3}}\left(\frac{1}{b}+\frac{1}{c}\right)$$

Proof. We have:

$$w_b+w_c\leq m_b+m_c\overset{(1)}{\leq}\frac{a^2+b^2+c^2}{2\sqrt{3}b}+\frac{a^2+b^2+c^2}{2\sqrt{3}c}=\frac{a^2+b^2+c^2}{2\sqrt{3}}\left(\frac{1}{b}+\frac{1}{c}\right)$$

where (1) it follows from:

Lemma. In ΔABC the following relationship holds:

$$a\cdot m_a\leq\frac{a^2+b^2+c^2}{2\sqrt{3}}$$

Proof. Using $m_a^2=\frac{2b^2+2c^2-a^2}{4}$, we have: $a\cdot m_a\leq\frac{1}{2\sqrt{3}}\sum_{cyc}a^2\Leftrightarrow a^2\cdot m_a^2\leq\frac{1}{12}\left(\sum_{cyc}a^2\right)^2\Leftrightarrow$

$$a^2\cdot\frac{2b^2+2c^2-a^2}{4}\leq\frac{1}{12}\left(\sum_{cyc}a^4+2\sum_{cyc}b^2c^2\right)\Leftrightarrow$$

$$3a^2(2b^2+2c^2-a^2)\leq a^4+b^4+c^4+2b^2c^2+2c^2a^2+2a^2b^2\Leftrightarrow$$

$$4a^4+b^4+c^4+2b^2c^2-4a^2b^2-4a^2c^2\geq 0\Leftrightarrow(2a^2-b^2-c^2)^2\geq 0$$

Equality holds for $2a^2-b^2-c^2=0\Leftrightarrow 2a^2=b^2+c^2$.

Let's get back to the main problem. Using Lemma we get:

$$LHS=\sum_{cyc}\frac{w_b+w_c}{a^2(b+c)}\overset{Lemma}{\leq}\sum_{cyc}\frac{\frac{a^2+b^2+c^2}{2\sqrt{3}}\left(\frac{1}{b}+\frac{1}{c}\right)}{a^2(b+c)}=\frac{a^2+b^2+c^2}{2\sqrt{3}abc}\sum_{cyc}\frac{1}{a}\overset{Leuberger}{\leq}$$

$$\overset{Leuberger}{\leq}\frac{a^2+b^2+c^2}{2\sqrt{3}abc}\cdot\frac{s}{3Rr}\overset{Leibniz}{\leq}\frac{9R^2}{2\sqrt{3}\cdot 4Rrs}=\frac{\sqrt{3}}{8r^2}\overset{Euler}{\leq}\frac{\sqrt{3}}{16}\cdot\frac{R}{r^3}=RHS$$

Equality holds if and only if triangle is equilateral. For LHS, we have:

$$\sum_{cyc}\frac{w_b+w_c}{a^2(b+c)}\geq\sum_{cyc}\frac{h_b+h_c}{a^2(b+c)}=\sum_{cyc}\frac{\frac{2F}{b}+\frac{2F}{c}}{a^2(b+c)}=2F\sum_{cyc}\frac{\frac{1}{a}+\frac{1}{b}}{a^2(b+c)}=$$

$$=2F\sum_{cyc}\frac{\frac{b+c}{bc}}{a^2(b+c)}=\frac{2F}{abc}\sum_{cyc}\frac{1}{a}=\frac{2F}{4RF}\sum_{cyc}\frac{1}{a}\overset{Leuberger}{\geq}\frac{1}{2R}\cdot\frac{\sqrt{3}}{R}=\frac{\sqrt{3}}{2R^2}$$

A SIMPLE PROOF FOR KY FAN'S INEQUALITY

By Daniel Sitaru-Romania

Abstract: *In this paper it's presented a simple proof for Ky Fan's inequality, a generalization and a few applications.*

KY FAN'S INEQUALITY

If $x_1,x_2\in\left(0,\frac{1}{2}\right)$ then:

$$\frac{x_1x_2}{(x_1+x_2)^2}\leq\frac{(1-x_1)(1-x_2)}{(1-x_1+1-x_2)^2};(1)$$

If $x_1,x_2\in\left(0,\frac{1}{2}\right)$ then:

$$\frac{x_1x_2x_3}{(x_1+x_2+x_3)^3}\leq\frac{(1-x_1)(1-x_2)(1-x_3)}{(1-x_1+1-x_2+1-x_3)^2};(2)$$

If $x_1,x_2,\dots,x_n\in\left(0,\frac{1}{2}\right),n\in\mathbb{N},n\geq 2$ then:

$$\frac{x_1x_2\cdot\ldots\cdot x_n}{(x_1+x_2+\cdots+x_n)^n}\leq\frac{(1-x_1)(1-x_2)\cdot\ldots\cdot(1-x_n)}{(1-x_1+1-x_2+\cdots+1-x_n)^n};(3)$$

Proof of (1):

Let be $f:\left(0,\frac{1}{2}\right)\to\mathbb{R}, f(x)=\log\left(\frac{1}{x}-1\right)$, then $f'(x)=\left(\frac{1}{x}-1\right)'\left(\frac{1}{x}-1\right)^{-1}=\frac{1}{x(x-1)}$

$$f'(x)=\frac{1}{x-1}-\frac{1}{x},\qquad f''(x)=-\frac{1}{(x-1)^2}+\frac{1}{x^2}=\frac{-2x+1}{x^2(x-1)^2}\leq 0$$

f −convex on $\left(0,\frac{1}{2}\right)$. By Jensen's inequality:

$$f\left(\frac{x_1+x_2}{2}\right)\leq\frac{1}{2}\left(f(x_1)+f(x_2)\right)$$

$$\log\left(\frac{1}{\frac{x_1+x_2}{2}}-1\right)\leq\frac{1}{2}\left(\log\left(\frac{1}{x_1}-1\right)+\log\left(\frac{1}{x_2}-1\right)\right)$$

$$2\log\left(\frac{2}{x_1+x_2}-1\right)\leq\log\left(\frac{1}{x_1}-1\right)\left(\frac{1}{x_2}-1\right)$$

$$\log\left(\frac{2-x_1-x_2}{x_1+x_2}\right)^2\leq\log\frac{(1-x_1)(1-x_2)}{x_1x_2}$$

$$\left(\frac{2-x_1-x_2}{x_1x_2}\right)^2\leq\frac{(1-x_1)(1-x_2)}{x_1x_2}$$

$$\frac{x_1x_2}{(x_1+x_2)^2}\leq\frac{(1-x_1)(1-x_2)}{(1-x_1+1-x_2)^2}$$

Equality holds for $x_1=x_2$.

Proof for (2): By Jensen's inequality:

$$f\left(\frac{x_1+x_2+x_3}{3}\right)\leq\frac{1}{3}\left(f(x_1)+f(x_2)+f(x_3)\right)$$

$$\log\left(\frac{1}{\frac{x_1+x_2+x_3}{3}}-1\right)\leq\frac{1}{3}\left(\log\left(\frac{1}{x_1}-1\right)+\log\left(\frac{1}{x_2}-1\right)+\log\left(\frac{1}{x_3}-1\right)\right)$$

$$3\log\left(\frac{3}{x_1+x_2+x_3}-1\right)\leq\log\left(\frac{1}{x_1}-1\right)\left(\frac{1}{x_2}-1\right)\left(\frac{1}{x_3}-1\right)$$

$$\left(\frac{3-x_1-x_2-x_3}{x_1+x_2+x_3}\right)^3 \le \left(\frac{1-x_1}{x_1}\right)\left(\frac{1-x_2}{x_2}\right)\left(\frac{1-x_3}{x_3}\right)$$

$$\frac{x_1x_2x_3}{(x_1+x_2+x_3)^3} \le \frac{(1-x_1)(1-x_2)(1-x_3)}{(1-x_1+1-x_2+1-x_3)^3}$$

Equality holds for $x_1 = x_2 = x_3$.

Proof of (3): By Jensen's inequality:

$$f\left(\frac{x_1+x_2+\dots+x_n}{n}\right) \le \frac{1}{n}\left(f(x_1)+f(x_2)+\dots+f(x_n)\right)$$

$$\log\left(\frac{1}{\frac{x_1+x_2+\dots+x_n}{n}}-1\right) \le \frac{1}{n}\left(\log\left(\frac{1}{x_1}-1\right)+\log\left(\frac{1}{x_2}-1\right)+\dots+\log\left(\frac{1}{x_n}-1\right)\right)$$

$$n\log\left(\frac{n}{x_1+x_2+\dots+x_n}-1\right) \le \log\left[\left(\frac{1}{x_1}-1\right)\left(\frac{1}{x_2}-1\right)\cdot\ldots\cdot\left(\frac{1}{x_n}-1\right)\right]$$

$$\left(\frac{n-x_1-x_2-\dots-x_n}{x_1+x_2+\dots+x_n}\right)^n \le \left(\frac{1-x_1}{x_1}\right)\left(\frac{1-x_2}{x_2}\right)\cdot\ldots\cdot\left(\frac{1-x_n}{x_n}\right)$$

$$\frac{x_1x_2\dots x_n}{(x_1+x_2+\dots+x_n)^n} \le \frac{(1-x_1)(1-x_2)\cdot\ldots\cdot(1-x_n)}{(1-x_1+1-x_2+\dots+1-x_n)^n}$$

Equality holds for $x_1 = x_2 = \dots = x_n$.

GENERALIZATION FOR KY FAN'S INEQUALITY $(n=2)$

If $x_1, x_2 \in \left(0, \frac{1}{2}\right), \lambda_1, \lambda_2 > 0, \lambda_1+\lambda_2 = 1$ then:

$$\frac{x_1^{\lambda_1}\cdot x_2^{\lambda_2}}{\lambda_1x_1+\lambda_2x_2} \le \frac{(1-x_1)^{\lambda_1}\cdot(1-x_2)^{\lambda_2}}{\lambda_1(1-x_1)+\lambda_2(1-x_2)};(4)$$

Proof. By Jensen's inequality: $f(\lambda_1x_1+\lambda_2x_2) \le \lambda_1 f(x_1)+\lambda_2 f(x_2)$

$$\log\left(\frac{1}{\lambda_1x_1+\lambda_2x_2}-1\right) \le \lambda_1\log\left(\frac{1}{x_1}-1\right)+\lambda_2\log\left(\frac{1}{x_2}-1\right)$$

$$\log\left(\frac{1-(\lambda_1x_1+\lambda_2x_2)}{\lambda_1x_1+\lambda_2x_2}\right) \le \log\left(\frac{1}{x_1}-1\right)^{\lambda_1}\left(\frac{1}{x_2}-1\right)^{\lambda_2}$$

$$\frac{1-\lambda_1x_1-\lambda_2x_2}{\lambda_1x_1+\lambda_2x_2} \le \left(\frac{1-x_1}{x_1}\right)^{\lambda_1} \cdot \left(\frac{1-x_2}{x_2}\right)^{\lambda_2}$$

$$\frac{\lambda_1(1-x_1)+\lambda_2(1-x_2)}{\lambda_1x_1+\lambda_2x_2} \le \frac{(1-x_1)^{\lambda_1}\cdot(1-x_2)^{\lambda_2}}{x_1^{\lambda_1}\cdot x_2^{\lambda_2}}$$

$$\frac{x_1^{\lambda_1}\cdot x_2^{\lambda_2}}{\lambda_1x_1+\lambda_2x_2} \le \frac{(1-x_1)^{\lambda_1}\cdot(1-x_2)^{\lambda_2}}{\lambda_1(1-x_1)+\lambda_2(1-x_2)}$$

Equality holds for $x_1 = x_2$.

GENERALIZATION FOR KY FAN'S INEQUALITY $(\boldsymbol{n=3})$

If $\boldsymbol{x_1, x_2, x_3 \in \left(0, \frac{1}{2}\right), \lambda_1, \lambda_2, \lambda_3 > 0, \lambda_1+\lambda_2+\lambda_3 = 1}$ **then:**

$$\boldsymbol{\frac{x_1^{\lambda_1}\cdot x_2^{\lambda_2}\cdot x_3^{\lambda_3}}{\lambda_1x_1+\lambda_2x_2+\lambda_3x_3} \le \frac{(1-x_1)^{\lambda_1}\cdot(1-x_2)^{\lambda_2}\cdot(1-x_3)^{\lambda_3}}{\lambda_1(1-x_1)+\lambda_2(1-x_2)+\lambda_3(1-x_3)}}; (\boldsymbol{5})$$

Proof. By Jensen's inequality: $f(\lambda_1x_1+\lambda_2x_2+\lambda_3x_3) \le \lambda_1f(x_1)+\lambda_2f(x_2)+\lambda_3f(x_3)$

$$\log\left(\frac{1}{\lambda_1x_1+\lambda_2x_2+\lambda_3x_3}-1\right) \le \lambda_1\log\left(\frac{1}{x_1}-1\right)+\lambda_2\log\left(\frac{1}{x_2}-1\right)+\lambda_3\log\left(\frac{1}{x_3}-1\right)$$

$$\log\left(\frac{1-\lambda_1x_1-\lambda_2x_2-\lambda_3x_3}{\lambda_1x_1+\lambda_2x_2+\lambda_3x_3}\right) \le \log\left(\frac{1}{x_1}-1\right)^{\lambda_1}\left(\frac{1}{x_2}-1\right)^{\lambda_2}\left(\frac{1}{x_3}-1\right)^{\lambda_3}$$

$$\frac{1-\lambda_1x_1-\lambda_2x_2-\lambda_3x_3}{\lambda_1x_1+\lambda_2x_2+\lambda_3x_3} \le \left(\frac{1-x_1}{x_1}\right)^{\lambda_1}\cdot\left(\frac{1-x_2}{x_2}\right)^{\lambda_2}\cdot\left(\frac{1}{x_3}-1\right)^{\lambda_3}$$

$$\frac{\lambda_1(1-x_1)+\lambda_2(1-x_2)+\lambda_3(1-x_3)}{\lambda_1x_1+\lambda_2x_2+\lambda_3x_3} \le \frac{(1-x_1)^{\lambda_1}\cdot(1-x_2)^{\lambda_2}\cdot(1-x_3)^{\lambda_3}}{x_1^{\lambda_1}\cdot x_2^{\lambda_2}\cdot x_3^{\lambda_3}}$$

$$\frac{x_1^{\lambda_1}\cdot x_2^{\lambda_2}\cdot x_3^{\lambda_3}}{\lambda_1x_1+\lambda_2x_2+\lambda_3x_3} \le \frac{(1-x_1)^{\lambda_1}\cdot(1-x_2)^{\lambda_2}\cdot(1-x_3)^{\lambda_3}}{\lambda_1(1-x_1)+\lambda_2(1-x_2)+\lambda_3(1-x_3)}$$

Equality holds for $x_1 = x_2 = x_3$.

GENERALIZATION FOR KY FAN'S INEQUALITY $(\boldsymbol{n \in \mathbb{N}, n \ge 2})$

If $\boldsymbol{x_1, x_2, \dots, x_n \in \left(0, \frac{1}{2}\right), \lambda_1, \lambda_2, \dots, \lambda_n > 0, \lambda_1 + \lambda_2 + \cdots + \lambda_n = 1}$ then:

$$\frac{x_1^{\lambda_1} \cdot x_2^{\lambda_2} \cdot \dots \cdot x_n^{\lambda_n}}{\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n} \leq \frac{(1-x_1)^{\lambda_1} \cdot (1-x_2)^{\lambda_2} \cdot \dots \cdot (1-x_n)^{\lambda_n}}{\lambda_1(1-x_1) + \lambda_2(1-x_2) + \cdots + \lambda_n(1-x_n)}; (6)$$

Proof. By Jensen's inequality:

$$f(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2) + \cdots + \lambda_n f(x_n)$$

$$\log\left(\frac{1}{\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n} - 1\right) \leq \lambda_1 \log\left(\frac{1}{x_1} - 1\right) + \lambda_2 \log\left(\frac{1}{x_2} - 1\right) + \cdots + \lambda_n \log\left(\frac{1}{x_n} - 1\right)$$

$$\log\left(\frac{1 - \lambda_1 x_1 - \lambda_2 x_2 - \cdots - \lambda_n x_n}{\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n}\right) \leq \log\left[\left(\frac{1}{x_1} - 1\right)^{\lambda_1} \cdot \left(\frac{1}{x_2} - 1\right)^{\lambda_2} \cdot \dots \cdot \left(\frac{1}{x_n} - 1\right)^{\lambda_n}\right]$$

$$\frac{1 - \lambda_1 x_1 - \lambda_2 x_2 - \cdots - \lambda_n x_n}{\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n} \leq \left(\frac{1 - x_1}{x_1}\right)^{\lambda_1} \cdot \left(\frac{1 - x_2}{x_2}\right)^{\lambda_2} \cdot \dots \cdot \left(\frac{1}{x_n} - 1\right)^{\lambda_n}$$

$$\frac{\lambda_1(1-x_1) + \lambda_2(1-x_2) + \cdots + \lambda_n(1-x_n)}{\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n} \leq \frac{(1-x_1)^{\lambda_1} \cdot (1-x_2)^{\lambda_2} \cdot \dots \cdot (1-x_n)^{\lambda_n}}{x_1^{\lambda_1} \cdot x_2^{\lambda_2} \cdot \dots \cdot x_n^{\lambda_n}}$$

$$\frac{x_1^{\lambda_1} \cdot x_2^{\lambda_2} \cdot \dots \cdot x_n^{\lambda_n}}{\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n} \leq \frac{(1-x_1)^{\lambda_1} \cdot (1-x_2)^{\lambda_2} \cdot \dots \cdot (1-x_n)^{\lambda_n}}{\lambda_1(1-x_1) + \lambda_2(1-x_2) + \cdots + \lambda_n(1-x_n)}$$

Equality holds for $x_1 = x_2 = \cdots = x_n$.

Application 1. If $\boldsymbol{a, b \in \left(0, \frac{1}{2}\right), x \in \mathbb{R}}$ then:

$$\frac{a^{\sin^2 x} \cdot b^{\cos^2 x}}{a \sin^2 x + b \cos^2 x} \leq \frac{(1-a)^{\sin^2 x} \cdot (1-b)^{\cos^2 x}}{(1-a)\sin^2 x + (1-b)\cos^2 x}$$

Solution: We take in (4): $\lambda_1 = \sin^2 x, \lambda_2 = \cos^2 x, \lambda_1 + \lambda_2 = \sin^2 x + \cos^2 x = 1$

Application 2. If $\boldsymbol{a, b, c \in \left(0, \frac{1}{2}\right)}$ then:

$$\left(\frac{a}{1-a}\right)^3 \cdot \left(\frac{b}{1-b}\right)^2 \cdot \frac{c}{1-c} \leq \left(\frac{3a + 2b + c}{6 - 3a - 2b - c}\right)^6$$

Solution: We take in (5):

$$\lambda_1=\frac{1}{2},\lambda_2=\frac{1}{3},\lambda_3=\frac{1}{6},\lambda_1+\lambda_2+\lambda_3=\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1$$

$$\frac{a^{\frac{1}{2}}\cdot b^{\frac{1}{3}}\cdot c^{\frac{1}{6}}}{\frac{a}{2}+\frac{b}{3}+\frac{c}{6}}\le\frac{(1-a)^{\frac{1}{2}}\cdot(1-b)^{\frac{1}{3}}\cdot(1-c)^{\frac{1}{6}}}{\frac{1}{2}(1-a)+\frac{1}{3}(1-b)+\frac{1}{6}(1-c)}$$

$$\frac{\sqrt[6]{a^3b^3c^3}}{\sqrt[6]{(1-a)^3(1-b)^2(1-c)}}\le\frac{\frac{a}{2}+\frac{b}{3}+\frac{c}{6}}{1-\frac{a}{2}-\frac{b}{3}-\frac{c}{6}}$$

$$\frac{a^3b^2c}{(1-a)^3(1-b)^2(1-c)}\le\left(\frac{3a+2b+c}{6-3a-2b-c}\right)^6$$

$$\left(\frac{a}{1-a}\right)^3\cdot\left(\frac{b}{1-b}\right)^2\cdot\frac{c}{1-c}\le\left(\frac{3a+2b+c}{6-3a-2b-c}\right)^6$$

Equality holds for $a=b=c$.

REFERENCE: ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

ABOUT A FEW INEQUALITIES IN TRIANGLE

By D.M. Bătinețu-Giurgiu, Mihaly Bencze, Florică Anastase-Romania

In [1]. Arkady M. Alt has proved the following inequality:

$$(\boldsymbol{a^2+2})(\boldsymbol{b^2+2})(\boldsymbol{c^2+2})\ge\boldsymbol{9(ab+bc+ca)};\forall\boldsymbol{a,b,c}>0;\ \ (\boldsymbol{1})$$

In ΔABC with F −area, holds:

$$(\boldsymbol{a^2+2})(\boldsymbol{b^2+2})(\boldsymbol{c^2+2})\ge\boldsymbol{36\sqrt{3}\cdot F};\ \ (\boldsymbol{2})$$

Proof. If in (1) a,b,c −are the lengths sides of a triangle ABC with F −area, the following relationship holds:

$$(a^2+2)(b^2+2)(c^2+2)\ge 9(ab+bc+ca)\overset{Gordon}{\ge}9\cdot4\sqrt{3}F=36\sqrt{3}\cdot F$$

Theorem 1.

If $x,y,z>0$ then in ΔABC with F −area, the following relationship holds:

$$\left(\frac{\boldsymbol{x^2a^4}}{(\boldsymbol{y+z})^2}+\boldsymbol{1}\right)\left(\frac{\boldsymbol{y^2b^4}}{(\boldsymbol{z+x})^2}+\boldsymbol{1}\right)\left(\frac{\boldsymbol{z^2c^4}}{(\boldsymbol{x+y})^2}+\boldsymbol{1}\right)\ge\boldsymbol{36\cdot F^2};\ \ (*)$$

Theorem 2.

If $x, y, z > 0$ then in ΔABC with F −area, the following relationship holds:

$$\left(\frac{x^2a^8}{(y+z)^2}+1\right)\left(\frac{y^2b^8}{(z+x)^2}+1\right)\left(\frac{z^2c^8}{(x+y)^2}+1\right) \geq 48 \cdot F^4; \quad (**)$$

To prove the above inequality, first, we prove the next Lemma:

Lemma: If $u, v, w > 0$, then the following relationship holds:

$$(u^2+1)(v^2+1)(w^2+1) \geq \frac{3}{4}(u+v+w)^2; \quad (3)$$

Proof. We have:

$$(i): (u^2+1)(v^2+1) \geq (u+v)^2 \Leftrightarrow u^2v^2+u^2+v^2+1 \geq u^2+2uv+v^2$$

$$u^2v^2-2uv+1 \geq 0 \Leftrightarrow (uv-1)^2 \geq 0 \text{ true!}$$

$$(ii): (u^2+1)(v^2+1) \geq \frac{3}{4}((u+v)^2+1) \Leftrightarrow$$

$$u^2v^2+u^2+v^2+1 \geq \frac{3}{4}(u^2+2uv+v^2+1) \Leftrightarrow$$

$$4u^2v^2+4u^2+4v^2+4 \geq 3u^2+3v^2+6uv+3 \Leftrightarrow$$

$$4u^2v^2-4uv+1+u^2-2uv+v^2 \geq 0 \Leftrightarrow (2uv-1)^2+(u-v)^2 \geq 0 \text{ true!}$$

Hence, we have:

$$(u^2+1)(v^2+1)(w^2+1) \overset{(ii)}{\geq} \frac{3}{4}((u+v)^2+1)(w^2+1) \overset{(i)}{\geq} \frac{3}{4}\big((u+v)+w\big)^2 \Leftrightarrow$$

$$(u^2+1)(v^2+1)(w^2+1) \geq \frac{3}{4}(u+v+w)^2$$

Proof of Theorem 1.

In Lemma, we take: $u = \frac{xa^2}{y+z}, v = \frac{yb^2}{z+x}, w = \frac{zc^2}{x+y}$, we get:

$$\prod_{cyc}\left(\frac{x^2a^4}{(y+z)^2}+1\right) \geq \frac{3}{4}\left(\sum_{cyc}\frac{x}{y+z}\cdot a^2\right)^2 \overset{Tsintsifas}{\geq}$$

$$\geq \frac{3}{4}\big(4\sqrt{3}\cdot F\big)^2 = \frac{3\cdot 16\cdot 3\cdot F^2}{4} = 36\cdot F^2$$

Proof of Theorem 2.

If in Lemma, we take: $u=\frac{xa^4}{y+z}, v=\frac{yb^4}{z+x}, w=\frac{zc^4}{x+y}$, we get:

$$\prod_{cyc}\left(\frac{x^2a^8}{(y+z)^2}+1\right)\geq\frac{3}{4}\left(\sum_{cyc}\frac{ax^4}{y+z}\right)^2=\frac{3}{4}\left(\sum_{cyc}\frac{x^2x^4}{xy+xz}\right)^2\overset{Bergstrom}{\geq}$$

$$\geq\frac{3}{4}\left(\frac{xa^2+yb^2+zc^2}{2(xy+yz+zx)}\right)^2=\frac{3}{4}\frac{(xa^2+yb^2+zc^2)^4}{4\big(2(xy+yz+zx)\big)^2}=$$

$$=\frac{3}{16}\frac{(xa^2+yb^2+zc^2)^4}{(xy+yz+zx)^2}\overset{Oppenheim}{\geq}\frac{3(16(xy+yz+zx)F^2)^2}{16(xy+yz+zx)^2}=$$

$$=3\cdot16\cdot F^4=48\cdot F^4$$

REFERENCE:

[1]. **Alt M. Arkady,** ABOUT ONE INEQUALITY FROM A.P.M.O. 2004-NEW SOLUTION AND GENERALIZATIONS-Octogon Mathematical Magazine, Vol.27, No.1, April 2019,pag.228-232.

ABOUT AN INEQUALITY BY GEORGE APOSTOLOPOULOS-XI

By Marin Chirciu-Romania

1) In ΔABC the following inequality holds:

$$\sum r_ah_a\tan\frac{A}{2}\leq F\left(\frac{2R}{r}-1\right)$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution: We prove: **Lemma:**

$$\sum r_ah_a\tan\frac{A}{2}=\frac{p^2(r^2-8Rr)+r(4R+r)^3}{2pR}$$

Proof: We have:

$$\sum r_ah_a\tan\frac{A}{2}=\sum\frac{S}{s-a}\cdot\frac{2S}{a}\cdot\sqrt{\frac{(p-b)(p-c)}{p(p-a)}}=$$

$$=2S^2\sum\frac{1}{a(p-a)}\cdot\frac{\sqrt{p(p-a)(p-b)(p-c)}}{p(p-a)}=$$

$$= 2S^2 \cdot \frac{S}{p}\sum\frac{1}{a(p-a)}\cdot\frac{1}{(p-a)} = 2p^2r^2\cdot\frac{pr}{p}\sum\frac{1}{a(p-a)^2} = 2p^2r^3\sum\frac{1}{a(p-a)^2} =$$

$= 2p^2r^3 \cdot \frac{p^2(r-8R)+(4R+r)^3}{4Rr^2p^3} = \frac{p^2(r^2-8Rr)+r(4R+r)^3}{2pR}$, which follows from:

$$\sum\frac{1}{a(p-a)^2} = \frac{\sum bc(p-b)^2(p-c)^2}{abc\cdot\prod(p-a)^2} = \frac{r^3[p^2(r-8R)+(4R+r)^3]}{4Rrp\cdot(r^2p)^2} =$$

$= \frac{p^2(r-8R)+(4R+r)^3}{4Rr^2p^3}$, true from:

$$\sum \boldsymbol{bc(p-b)^2(p-c)^2 = r^3[p^2(r-8R)+(4R+r)^3]}$$

Let's get to the main problem.Using Lemma the inequality can be written:

$$\frac{p^2(r^2-8Rr)+r(4R+r)^3}{2pR} \le F\left(\frac{2R}{r}-1\right) \Leftrightarrow \frac{p^2(r^2-8Rr)+r(4R+r)^3}{2pR} \le$$

$$\le pr\left(\frac{2R-r}{r}\right) \Leftrightarrow \frac{p^2(r^2-8Rr)+r(4R+r)^3}{2pR} \le pr\left(\frac{2R-r}{r}\right) \Leftrightarrow$$

$$\Leftrightarrow p^2(r^2-8Rr)+r(4R+r)^3 \le 2p^2(2R-r) \Leftrightarrow$$

$\Leftrightarrow p^2(4R^2+6Rr-r^2) \ge r(4R+r)^3$, which follows from Gerretesen's inequality:

$p^2 \ge 16Rr - 5r^2 \ge \frac{r(4R+r)^2}{R+r}$. It remains to prove that:

$$\frac{r(4R+r)^2}{R+r}(4R^2+6Rr-r^2) \ge r(4R+r)^3 \Leftrightarrow (4R^2+6Rr-r^2) \ge$$

$\ge (R+r)(4R+r) \Leftrightarrow R \ge 2r$, (Euler's inequality)

Equality holds if and only if the triangle is equilateral.**Remark:** The inequality can be strengthened.

2) In ΔABC the following inequality holds:

$$\sum \boldsymbol{r_a h_a \tan\frac{A}{2} \le F\left(\frac{2R}{r}+\frac{r}{R}-\frac{3}{2}\right)}$$

Marin Chirciu

Proof: Using the Lemma the inequality can be written:

$$\frac{p^2(r^2-8Rr)+r(4R+r)^3}{2pR} \le F\left(\frac{2R}{r}+\frac{r}{R}-\frac{3}{2}\right) \Leftrightarrow$$

$$\Leftrightarrow \frac{p^2(r^2-8Rr)+r(4R+r)^3}{2pR} \le pr\left(\frac{4R^2-3Rr+2r^2}{2Rr}\right) \Leftrightarrow$$

$$\Leftrightarrow p^2(r^2-8Rr)+r(4R+r)^3\le p^2(4R^2-3Rr+2r^2)\Leftrightarrow$$

$$\Leftrightarrow p^2(4R^2+5Rr+r^2)\ge r(4R+r)^3\Leftrightarrow$$

$$\Leftrightarrow p^2(4R+r)(R+r)\ge r(4R+r)^3\Leftrightarrow p^2(R+r)\ge r(4R+r)^2\Leftrightarrow p^2\ge\frac{r(4R+r)^2}{R+r},$$

which follows from Gerretsen's inequality: $p^2\ge 16Rr-5r^2\ge\frac{r(4R+r)^2}{R+r}$

Equality holds if and only if the triangle is equilateral. **Remark:** Inequality 2) is stronger than inequality 1)

3) In ΔABC the following relationship holds:

$$\sum \boldsymbol{r_a h_a \tan\frac{A}{2}\le F\left(\frac{2R}{r}+\frac{r}{R}-\frac{3}{2}\right)\le F\left(\frac{2R}{r}-1\right)}$$

Solution: See inequality 1) and $F\left(\frac{2R}{r}+\frac{r}{R}-\frac{3}{2}\right)\le F\left(\frac{2R}{r}-1\right)\Leftrightarrow R\ge 2r$, (Euler's inequality).

Equality holds if and only if the triangle is equilateral.**Remark:** Let's find an inequality having an opposite sense.

4) In ΔABC the following relationship holds:

$$\sum \boldsymbol{r_a h_a \tan\frac{A}{2}\ge 3F}$$

Marin Chirciu

Proof: Using the Lemma we obtain:

$$\sum r_a h_a\tan\frac{A}{2}=\frac{p^2(r^2-8Rr)+r(4R+r)^3}{2pR}=\frac{p}{2R}\left[(r^2-8Rr)+\frac{r(4R+r)^3}{p^2}\right]\ge$$

$$\overset{Gerretsen}{\ge}\frac{p}{2R}\left[(r^2-8Rr)+\frac{r(4R+r)^3}{\frac{R(4R+r)^2}{2(2R-r)}}\right]=\frac{p}{2R}\left[(r^2-8Rr)+\frac{2r(2R-r)(4R+r)}{R}\right]=$$

$$=\frac{pr}{2R}\left[\frac{R(r-8R)+2(2R-r)(4R+r)}{R}\right]=\frac{S}{2R}\cdot\frac{8R^2-3Rr-2r^2}{R}\overset{Euler}{\ge}\frac{S}{2R}\cdot 6R=3S$$

Equality holds if and only if the triangle is equilateral.**Remark:** We can write the double inequality:

5) In ΔABC the following inequality holds:

$$\boldsymbol{3F\le\sum r_a h_a\tan\frac{A}{2}\le F\left(\frac{2R}{r}+\frac{r}{R}-\frac{3}{2}\right)}$$

Marin Chirciu

Proof: RHS inequality.Using the Lemma the inequality can be written:

$$\frac{p^2(r^2-8Rr)+r(4R+r)^3}{2pR}\le F\left(\frac{2R}{r}+\frac{r}{R}-\frac{3}{2}\right)\Leftrightarrow$$

$$\Leftrightarrow\frac{p^2(r^2-8Rr)+r(4R+r)^3}{2pR}\le pr\left(\frac{4R^2-3Rr+2r^2}{2Rr}\right)\Leftrightarrow$$

$$\Leftrightarrow p^2(r^2-8Rr)+r(4R+r)^3\le p^2(4R^2-3Rr+2r^2)\Leftrightarrow$$

$$\Leftrightarrow p^2(4R^2+5Rr+r^2)\ge r(4R+r)^3\Leftrightarrow$$

$$\Leftrightarrow p^2(4R+r)(R+r)\ge r(4R+r)^3\Leftrightarrow p^2(R+r)\ge r(4R+r)^2\Leftrightarrow p^2\ge\frac{r(4R+r)^2}{R+r},$$

which follows from Gerretsen's inequality: $p^2\ge 16Rr-5r^2\ge\frac{r(4R+r)^2}{R+r}$.

Equality holds if and only if the triangle is equilateral. LHS inequality. Using the Lemma we obtain:

$$\sum r_a h_a\tan\frac{A}{2}=\frac{p^2(r^2-8Rr)+r(4R+r)^3}{2pR}=\frac{p}{2R}\left[(r^2-8Rr)+\frac{r(4R+r)^3}{p^2}\right]\ge$$

$$\overset{Gerretsen}{\ge}\frac{p}{2R}\left[(r^2-8Rr)+\frac{r(4R+r)^3}{\frac{R(4R+r)^2}{2(2R-r)}}\right]=\frac{p}{2R}\left[(r^2-8Rr)+\frac{2r(2R-r)(4R+r)}{R}\right]=$$

$$=\frac{pr}{2R}\left[\frac{R(r-8R)+2(2R-r)(4R+r)}{R}\right]=\frac{S}{2R}\cdot\frac{8R^2-3Rr-2r^2}{R}\overset{Euler}{\ge}\frac{S}{2R}\cdot 6R=3S$$

Equality holds if and only if the triangle is equilateral.

Reference: ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

ABOUT LOGARITHMIC MEAN AND APPLICATIONS

By Daniel Sitaru-Romania

Abstract: *In this article is revisited logarithmic mean, it's properties and a few applications.*

Definition

Let $a,b>0$. We define:

$$L(a,b)=\begin{cases}\dfrac{b-a}{\ln b-\ln a}; a\ne b\\ a; a=b\end{cases}$$

and name this the logarithmic mean of a,b.

Property 1.

If $0 < a < b$ then:

$$\boldsymbol{L(a,b) = \left(\int_0^1 \frac{dx}{xa+(1-x)b}\right)^{-1}}$$

Proof.

$$\int_0^1 \frac{dx}{xa+(1-x)b} = \int_0^1 \frac{dx}{x(a-b)+b} = \frac{1}{a-b}\int_0^1 \frac{(x(a-b)+b)'}{x(a-b)+b}dx =$$

$$= \frac{1}{a-b}\ln|(a-b)x+b||_0^1 = \frac{1}{a-b}(\ln a - \ln b) = \frac{\ln b - \ln a}{b-a}$$

$$\left(\int_0^1 \frac{dx}{xa+(1-x)b}\right)^{-1} = \frac{b-a}{\ln b - \ln a} = L(a,b)$$

Property 2.

If $0 < a < b$ then:

$$\boldsymbol{L(a,b) = \left(\int_0^\infty \frac{dx}{(x+a)(x+b)}\right)^{-1}}$$

Proof.

$$\int_0^\infty \frac{dx}{(x+a)(x+b)} = \frac{1}{a-b}\int_0^\infty \frac{x+a-(x+b)}{(x+a)(x+b)}dx =$$

$$= \frac{1}{a-b}\left(\int_0^\infty \left(\frac{x+a}{(x+a)(x+b)} - \frac{x+b}{(x+a)(x+b)}\right)dx\right) =$$

$$= \frac{1}{a-b}\left(\int_0^\infty \frac{dx}{x+b} - \int_0^\infty \frac{dx}{x+a}\right) = \frac{1}{a-b}\ln\left(\frac{x+b}{x+a}\right)\Bigg|_0^\infty =$$

$$= \frac{1}{a-b}\left(\ln 1 - \ln\left(\frac{b}{a}\right)\right) = \frac{\ln a - \ln b}{a-b} = \frac{\ln b - \ln a}{b-a}$$

$$\left(\int_0^\infty \frac{dx}{(x+a)(x+b)}\right)^{-1} = \left(\frac{\ln b - \ln a}{b-a}\right)^{-1} = \frac{b-a}{\ln b - \ln a} = L(a,b)$$

Property 3.

If $0 < a < b$ then:

$$\boldsymbol{L(a,b) < \frac{a+b}{2}}$$

Proof 1. Let be $f:[1,\infty)\to\mathbb{R}; f(x)=\ln x-\frac{2(x-1)}{x+1}$, then

$$f'(x)=\frac{1}{x}-\frac{2(x+1)-2(x-1)}{(x+1)^2}=\frac{(x+1)^2-4x}{x(x+1)^2}=\frac{(x-1)^2}{x(x+1)^2}>0$$

$$\min f(x)=f(1)=0\Rightarrow f(x)\geq 0; (\forall)x\geq 1$$

$$\ln x\geq\frac{2(x-1)}{x+1};\ \ (1)$$

If $a,b>0; a<b\Rightarrow\frac{b}{a}>1$. Replace $x=\frac{b}{a}$ in (1):

$$\ln\left(\frac{b}{a}\right)>\frac{2\left(\frac{b}{a}-1\right)}{\frac{b}{a}+1},\qquad \ln b-\ln a>\frac{2(b-a)}{b+a}$$

$$\frac{1}{\ln b-\ln a}<\frac{a+b}{2(b-a)},\qquad \frac{b-a}{\ln b-\ln a}<\frac{a+b}{2}$$

$$L(a,b)<\frac{a+b}{2}$$

Proof 2. For $x\geq 0$ we have:

$$(x+a)(x+b)=x^2+(a+b)x+ab<x^2+(a+b)x+ab+\left(\frac{a-b}{2}\right)^2=$$

$$=x^2+(a+b)x+\left(\frac{a+b}{2}\right)^2=\left(x+\frac{a+b}{2}\right)^2$$

$$(x+a)(x+b)<\left(x+\frac{a+b}{2}\right)^2,\qquad \frac{1}{(x+a)(x+b)}>\frac{1}{\left(x+\frac{a+b}{2}\right)^2}$$

$$\int_0^\infty\frac{dx}{(x+a)(x+b)}>\int_0^\infty\frac{dx}{\left(x+\frac{a+b}{2}\right)^2}$$

By property 2:

$$\left(L(a,b)\right)^{-1}>-\frac{1}{x+\frac{a+b}{2}}\Bigg|_0^\infty=\frac{1}{\frac{a+b}{2}},\qquad \left(L(a,b)\right)^{-1}>\frac{2}{a+b},\ \ L(a,b)<\frac{a+b}{2}$$

Property 4.

If $0<a<b$ then:

$$\boldsymbol{L(a,b)>\sqrt{ab}}$$

Proof 1. Let be $g:[1,\infty)\to\mathbb{R}; g(x)=x-\frac{1}{x}-2\ln x$, then

$$g'(x)=1+\frac{1}{x^2}-\frac{2}{x}=\frac{x^2+1-2x}{x^2}=\frac{(x-1)^2}{x^2}\geq 0$$

$$g(x)\geq g(1)=0\Rightarrow x-\frac{1}{x}-2\ln x\geq 0$$

$$2\ln x\leq x-\frac{1}{x};\ \ (2)$$

If $a,b>0; b>a\Rightarrow\sqrt{\frac{b}{a}}>1$. Replace $x=\sqrt{\frac{b}{a}}$ in (2):

$$2\ln\left(\sqrt{\frac{b}{a}}\right)<\sqrt{\frac{b}{a}}-\sqrt{\frac{a}{b}},\qquad 2\cdot\frac{1}{2}\ln\left(\frac{b}{a}\right)<\sqrt{\frac{b}{a}}-\sqrt{\frac{a}{b}}$$

$$\ln b-\ln a<\frac{b-a}{\sqrt{ab}},\qquad \frac{\ln b-\ln a}{b-a}<\frac{1}{\sqrt{ab}}$$

$$\sqrt{ab}<\frac{b-a}{\ln b-\ln a},\ \ L(a,b)>\sqrt{ab}$$

Proof 2. If $x\geq 0$ then:

$$(x+a)(x+b)=x^2+(a+b)x+ab\overset{AM-GM}{>}x^2+2\sqrt{ab}x+ab=$$

$$=\left(x+\sqrt{ab}\right)^2$$

$$(x+a)(x+b)>\left(x+\sqrt{ab}\right)^2,\qquad \frac{1}{(x+a)(x+b)}<\frac{1}{\left(x+\sqrt{ab}\right)^2}$$

$$\int_0^\infty\frac{dx}{(x+a)(x+b)}<\int_0^\infty\frac{dx}{\left(x+\sqrt{ab}\right)^2}$$

By property 2:

$$\left(L(a,b)\right)^{-1}<-\frac{1}{x+\sqrt{ab}}\bigg|_0^\infty=\frac{1}{\sqrt{ab}},\quad \left(L(a,b)\right)^{-1}<\frac{1}{\sqrt{ab}}$$

$$L(a,b)>\sqrt{ab}$$

Property 5.

If $a,b>0, a\neq b$ then:

$$\frac{L(a^2,b^2)}{L(a,b)}=\frac{b+a}{2}$$

Proof.

$$\frac{L(a^2,b^2)}{L(a,b)}=\frac{b^2-a^2}{\ln b^2-\ln a^2}\cdot\left(\frac{b-a}{\ln b-\ln a}\right)^{-1}=\frac{(b-a)(b+a)}{2(\ln b-\ln a)}\cdot\frac{\ln b-\ln a}{b-a}=\frac{b+a}{2}$$

Property 6.

If $\boldsymbol{a},\boldsymbol{b}>0, a\neq b$ then:

$$\sqrt{\frac{\boldsymbol{L(a,b)}}{\boldsymbol{L\left(\frac{1}{a},\frac{1}{b}\right)}}}=\sqrt{\boldsymbol{ab}}$$

Proof.

$$\sqrt{\frac{L(a,b)}{L\left(\frac{1}{a},\frac{1}{b}\right)}}=\sqrt{\frac{b-a}{\ln b-\ln a}\cdot\left(\frac{\frac{1}{b}-\frac{1}{a}}{\ln\frac{1}{b}-\ln\frac{1}{a}}\right)^{-1}}=\sqrt{\frac{b-a}{\ln b-\ln a}\cdot\frac{\ln a-\ln b}{a-b}\cdot ab}=\sqrt{ab}$$

Property 7.

If $\boldsymbol{a},\boldsymbol{b}>0, a\neq b$ then:

$$\frac{\boldsymbol{L\left(\frac{1}{a},\frac{1}{b}\right)}}{\boldsymbol{L\left(\frac{1}{a^2},\frac{1}{b^2}\right)}}=\frac{\mathbf{2}}{\frac{\mathbf{1}}{\boldsymbol{a}}+\frac{\mathbf{1}}{\boldsymbol{b}}}$$

Proof.

$$\frac{L\left(\frac{1}{a},\frac{1}{b}\right)}{L\left(\frac{1}{a^2},\frac{1}{b^2}\right)}=\frac{\frac{1}{b}-\frac{1}{a}}{\ln\frac{1}{b}-\ln\frac{1}{a}}\cdot\left(\frac{\frac{1}{b^2}-\frac{1}{a^2}}{\ln\frac{1}{b^2}-\ln\frac{1}{a^2}}\right)^{-1}=$$

$$=\frac{a-b}{ab}\cdot\frac{1}{\ln a-\ln b}\cdot\frac{2(\ln a-\ln b)}{a^2-b^2}\cdot a^2b^2=\frac{2(a-b)}{(a-b)(a+b)}\cdot ab=\frac{2ab}{a+b}=\frac{2}{\frac{1}{a}+\frac{1}{b}}$$

Property 8.

If $\boldsymbol{a},\boldsymbol{b}>0, a\neq b$ then:

$$\frac{\boldsymbol{L\left(\frac{1}{a},\frac{1}{b}\right)}}{\boldsymbol{L\left(\frac{1}{a^2},\frac{1}{b^2}\right)}}<\sqrt{\frac{\boldsymbol{L(a,b)}}{\boldsymbol{L\left(\frac{1}{a},\frac{1}{b}\right)}}}<\frac{\boldsymbol{L(a^2,b^2)}}{\boldsymbol{L(a,b)}}$$

Proof.

By $HM<GM<AM$: $\frac{2ab}{a+b}<\sqrt{ab}<\frac{a+b}{2}$ and by Properties 5,6,7:

$$\frac{L\left(\frac{1}{a},\frac{1}{b}\right)}{L\left(\frac{1}{a^2},\frac{1}{b^2}\right)} < \sqrt{\frac{L(a,b)}{L\left(\frac{1}{a},\frac{1}{b}\right)}} < \frac{L(a^2,b^2)}{L(a,b)}$$

Reference: ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

INEQUALITIES RELATED TO GENERALIZED HYPERBOLIC FUNCTIONS AND LOGARITHMIC MEAN

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Abstract: This article introduces some new inequalities related to generalized hyperbolic functions and logarithmic mean.

Keywords: Hiperbolic functions, logarithmic mean

1. INTRODUCTION

HUYGENS' INEQUALITY

$$2\frac{\sin x}{x}+\frac{\tan x}{x}>3, \qquad (1)$$

CUSA- HUYGENS INEQUALITY

$$\frac{\sin x}{x}<\frac{\cos x+2}{3} \qquad (2)$$

These inequalities are true for every $x \in \left(0,\frac{\pi}{2}\right)$.[1–6,] $sinh(x)=\frac{e^x-e^{-x}}{2}, cosh(x)=\frac{e^x+e^{-x}}{2}, tanh(x)=\frac{e^x-e^{-x}}{e^x+e^{-x}}$

These functions are called hyperbolic sine, cosine and tangent functions, respectively.Logarithmic mean has applications in mathematics and physics. In this article we will present a generalization of the logarithmic mean. The logarithmic mean of two positive numbers a and b is the number $L(a,b)$ defined as[7]

$$L(a,b)=\frac{a-b}{loga-logb}\ ,a\neq b\ ,$$

with the convention that:

$$L(a,a)=\lim_{b\to a}L(a,b)=a$$

2. PRELIMINARIES

DEFINITION. $sinh_{\varphi}(x) = \frac{\varphi^x - \varphi^{-x}}{2}$, $cosh_{\varphi}(x) = \frac{\varphi^x + \varphi^{-x}}{2}$, (3)

$$tanh_{\varphi}(x) = \frac{\varphi^x - \varphi^{-x}}{\varphi^x + \varphi^{-x}}, x \in R, \varphi > 1$$

These functions are called generalized hyperbolic sine, cosine, and tangent functions, respectively.[8]

LEMMA 1.[3] If $\forall \varphi > 1$ and $x \geq 0$ then the following inequality is satisfied:

$$sinh_{\varphi}(x) \geq x ln\varphi \quad (4)$$

Proof. Let f: $R^+ \to R$ be a function defined by $f(x) = sinh_{\varphi}(x) - x ln\varphi$

The derivative of f(x) is $f'(x) = ln\varphi\left(cosh_{\varphi}(x) - 1\right)$

If we apply the AM-GM inequality, we get that:

$cosh_{\varphi}(x) \geq 1$, for all $x \in R$.

Then we obtain : $f'(x) \geq 0$, for all $x \in R$.

This show that:$f(x), x \in R$ is an increasing function. Then we obtain

For all $x \geq 0$, $f(x) \geq f(0) = 0$. □

LEMMA 2.[3] ***If $\forall \varphi > 1$ and $x \geq 0$ then the following inequality is satisfied:***

$$tanh_{\varphi}(x) \leq x ln\varphi \quad (5)$$

Proof: Let $f: R^+ \to R$ be a function defined by $f(x) = tanh_{\varphi}(x) - x ln\varphi$

The derivative of f(x) is $f'(x) = ln\varphi\left(\frac{1}{cosh_{\varphi}^2} - 1\right) \leq 0$.

This show that $f(x), x \geq 0$ is an decreasing function. Then we obtain :

For all $x \geq 0$, $f(x) \leq f(0) = 0$. □

MAIN RESULT

THEOREM 1 ***If $x \geq 0$ and $s > f > 1$ then the following inequality is satisfied:***

$$sinh_s(x) \geq sinh_f(x) \quad (6)$$

Proof: Let f: $R^+ \to R$ be a function defined by

$$f(x) = sinh_s(x) - sinh_f(x)$$

$$f'(x) = ln\varphi\left(cosh_s(x) - cosh_f(x)\right) = \frac{s^x + s^{-x}}{2} - \frac{f^x + f^{-x}}{2}$$

Obviously for all $x \geq 0$, $(s^x - f^x)\left(1 - \frac{1}{s^x f^x}\right) \geq 0$.This show that:

$\frac{s^x+s^{-x}}{2}-\frac{f^x+f^{-x}}{2}\geq 0$. Then we obtain $f'(x)\geq 0$, for all $x\geq 0$.This show $f(x), x\geq 0$ is an increasing function. Then we obtain For all $x\geq 0$, $f(x)\geq f(0)=0$. □

THEOREM 2: ***If $x\in R$ and $s>f>1$ then the following inequality is satisfied:***

$$\boldsymbol{cosh_s(x)\geq cosh_f(x)} \tag{7}$$

Proof: Let f: $R^+\to R$ be a function defined by $f(x)=cosh_s(x)-cosh_f(x)$

$f(x)$ is an even function then enough to show that theorem is true for $x\geq 0$.

The derivative of $f(x)$ is $f'(x)=ln\varphi\left(sinh_s(x)-sinh_f(x)\right)$

if we use *Theorem 1*, then we obtain $f'(x)\geq 0$. This show $f(x), x\geq 0$ is an increasing function. Then we obtain for all $x\geq 0$, $f(x)\geq f(0)=0$.

THEOREM 3: a and b are two positive numbers and $a\neq b$ then the following inequality is satisfied:

$$\sqrt{\boldsymbol{ab}}<\frac{\boldsymbol{a-b}}{(\boldsymbol{log_\varphi a-log_\varphi b})\boldsymbol{ln\,\varphi}}<\frac{\boldsymbol{a+b}}{\boldsymbol{2}}, \boldsymbol{\varphi}>1 \tag{8}$$

Proof: From *Lemma* 2 we get that for all $0\neq z\in R$ the following inequality is true:

$$\frac{tanh_\varphi(z)}{zln\varphi}<1,$$

If we take : $z=\frac{x-y}{2}$ then $\frac{tanh_\varphi\left(\frac{x-y}{2}\right)}{\frac{x-y}{2}ln\varphi}<1$, this means that:

$$\frac{\varphi^{\frac{x-y}{2}}-\varphi^{\frac{y-x}{2}}}{\frac{x-y}{2}\left(\varphi^{\frac{x-y}{2}}+\varphi^{\frac{y-x}{2}}\right)ln\varphi}<1$$

$$\frac{\varphi^{\frac{x-y}{2}}-\varphi^{\frac{y-x}{2}}}{\frac{x-y}{2}\left(\varphi^{\frac{x-y}{2}}+\varphi^{\frac{y-x}{2}}\right)ln\varphi}=\frac{\varphi^x-\varphi^y}{\frac{x-y}{2}(\varphi^x+\varphi^y)ln\varphi}\cdot\frac{\varphi^{-\frac{y+x}{2}}}{\varphi^{-\frac{y+x}{2}}}<1$$

This show that $\frac{\varphi^x-\varphi^y}{\frac{x-y}{2}ln\varphi}<\varphi^x+\varphi^y$

If we take $a=\varphi^x, b=\varphi^y$ then we obtain: $\frac{a-b}{(log_\varphi a-log_\varphi b)ln\,\varphi}<\frac{a+b}{2}$.

From *Lemma 1* we get that for all $0\neq z\in R$ the following inequality is true:

$$\frac{sinh_\varphi(z)}{zln\varphi}>1,$$

If we take : $z=\frac{x-y}{2}$ then $\frac{sinh_\varphi\left(\frac{x-y}{2}\right)}{\frac{x-y}{2}ln\varphi}>1$, this means that:

$$\frac{\varphi^{\frac{x-y}{2}}-\varphi^{\frac{y-x}{2}}}{\frac{x-y}{2}ln\varphi}=\frac{\varphi^x-\varphi^y}{\frac{x-y}{2}ln\varphi}\varphi^{-\frac{y+x}{2}}>1, \text{ we obtain}$$

$\frac{\varphi^x-\varphi^y}{\frac{x-y}{2}ln\varphi}>\varphi^{\frac{y+x}{2}}$, Now we take $a=\varphi^x, b=\varphi^y$ then we obtain:

$$\frac{a-b}{(\,log_\varphi a-log_\varphi b)ln\,\varphi}>\sqrt{ab}\,.$$

The proof of theorem is complete. □

Theorem 4 and theorem 5 are proved in [8]. We will give a few different proofs:

THEOREM 4 (HUYGENS' INEQUALITY)*If $x\neq 0$ and $x\in R$ then the following inequality is satisfied:*

$$2\frac{sinh_\varphi(x)}{x}+\frac{tanh_\varphi(x)}{x}>3ln\varphi, \forall\varphi>1 \tag{9}$$

Proof: Let f: $R^+\to R$ be a function defined by

$$f(x)=2sinh_\varphi(x)+tanh_\varphi(x)-3xln\varphi, x\geq 0$$

$$f'(x)=\left(2cosh_\varphi(x)+\frac{1}{cosh_\varphi^2(x)}-3\right)ln\varphi$$

If we apply the AM-GM inequality, we get that: $f'(x)\geq 0$, for all $x\in R$.

Then we obtain $f(x), x\geq 0$ is an increasing function. This show

$\forall x\geq 0, f(x)\geq f(0)$. We obtain: $2sinh_\varphi(x)+tanh_\varphi(x)\geq 3xln\varphi,\ x\geq 0$

Now if we divide both sides of the inequality by $x>0$ then we obtain

$$2\frac{sinh_\varphi(x)}{x}+\frac{tanh_\varphi(x)}{x}>3ln\varphi.$$

The proof of theorem is complete. □

THEOREM 5(CUSA-HUYGENS INEQUALITY) ***If $x\neq 0$ and $x\in R$ then the following inequality is satisfied:***

$$\frac{sinh_\varphi(x)}{x}<\left[\frac{cosh_\varphi(x)+2}{3}\right]ln\varphi, \forall\varphi>1 \tag{10}$$

Proof: Let $f: R^+\to R$ be a function defined by

$$f(x)=x\left[cosh_\varphi(x)+2\right]ln\varphi-3sinh_\varphi(x), \qquad x\geq 0$$

$$f'(x)=2\left(1-cosh_\varphi(x)\right)ln\varphi+xsinh_\varphi(x)(ln\varphi)^2$$

$$f''(x)=(ln\varphi)^2\left(xcosh_\varphi(x)ln\varphi-sinh_\varphi(x)\right)$$

From *Lemma 2* we obtain that: $f''(x) \geq 0, x \geq 0$. This show that
$f'(x)$ and $f(x)$ are increasing and positive functions.

$$x\left[cosh_{\varphi}(x)+2\right]ln\varphi \geq 3sinh_{\varphi}(x)$$

Now if we divide both sides of the inequality by $x > 0$ then we obtain

$$\frac{sinh_{\varphi}(x)}{x} < \left[\frac{cosh_{\varphi}(x)+2}{3}\right]ln\varphi$$

The proof of theorem is complete. □

THEOREM 6: If $x \geq 0$ and $s > f > 1$ *then the following inequality is satisfied:*

a) $\boldsymbol{sinh_s(x)\ln f \geq sinh_f(x)\ln s}$ $\quad$ **(11)**

b) $\boldsymbol{tanh_s(x)\ln f \leq tanh_f(x)\ln s}$ $\quad$ **(12)**

c) $\boldsymbol{cosh_s(x)\ln f \geq cosh_f(x)\ln s}$ $\quad$ **(13)**

Proof: a) $f: R^+ \to R$ be a function defined by $f(x) = sinh_s(x)\ln f - sinh_f(x)\ln s$

$$f'(x) = \ln f \ln s\left(cosh_s(x) - cosh_f(x)\right)$$

From *Theorem 2* we obtain $f'(x) \geq 0$, for all $x \in R$. Then we obtain $f(x), x \geq 0$ is an increasing function. This show $f(x) \geq f(0)$ if $x \geq 0$.That means

$sinh_s(x)\ln f - sinh_f(x)\ln s \geq 0$.The proof of *part a)* is complete.

b) $f: R^+ \to R$ be a function defined by $f(x) = tanh_s(x)\ln f - tanh_f(x)\ln s$

$$f'(x) = \ln f \ln s\left(\frac{1}{cosh_s^2(x)} - \frac{1}{cosh_f^2(x)}\right)$$

From *Theorem 2* we obtain $f'(x) \leq 0$, for all $x \in R$. This show that $f(x), x \geq 0$ is an decreasing function. Then we obtain :

For all $x \geq 0$, $f(x) \leq f(0) = 0$.

$tanh_s(x)\ln f - tanh_f(x)\ln s \leq 0$.The proof of *part b)* is complete.

c) $f: R^+ \to R$ be a function defined by $f(x) = cosh_s(x)\ln f - cosh_f(x)\ln s$

$$f'(x) = \ln f \ln s\left(sinh_s(x) - sinh_f(x)\right)$$

From *Theorem 1* we obtain $f'(x) \geq 0$ for all $x \geq 0$. That means $f(x), x \geq 0$ is an increasing function. This show $f(x) \geq f(0)$ if $x \geq 0$. This means that

$cosh_s(x)\ln f - cosh_f(x)\ln s \geq 0$.The proof of theorem is complete.

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HUYGENS' AND WİLKER'S INEQUALITIES REVISITED

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ABSTRACT: This paper introduces some refinements of Huygens' inequality and a generalization of Wilker's inequality

Keywords: Huygens' inequality,Wilker's inequality

1. Introduction: Hugyens and Wilker's type inequalities are very useful in analysis. For example, we can say approximations of trigonometric functions with linear functions. More precise approximations can be made using the inequalities in this article.

The Huygens' inequality

$$\frac{2\sin x}{x}+\frac{\tan x}{x}>3, \qquad 0<x<\frac{\pi}{2}. \tag{1.1}$$

Or:

$$2\sin x+\tan x\geq 3x, \qquad 0\leq x<\frac{\pi}{2}. \tag{1.2}$$

The Mitrinovic-Adamovic's inequality

$$\frac{\sin x}{x}>\sqrt[3]{\cos x}, \qquad 0<x<\frac{\pi}{2}. \tag{1.3}$$

And Wilker's inequality

$$\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}>2 \qquad 0<x<\frac{\pi}{2}. \tag{1.4}$$

We can see the proof of these inequalities in [1-4].

2. Preliminiaries and some useful lemmas:

Lemma 1 [5]. (AM-GM inequality) If $a_i>0, i=\overline{1,n}$ then the following inequality is satisfied

$$\frac{\sum_{i=1}^n a_i}{n}\geq\left(\prod_{i=1}^n a_i\right)^{\frac{1}{n}}, \qquad n\in N^* \tag{2.1}$$

Lemma 2 [6]. If $a_i>0, i=\overline{1,n}$ then the following inequality is satisfied

$$\left(\sum_{i=1}^n (a_i+b_i)^2\right)\geq 4\left(\sum_{i=1}^n\left(\sqrt{a_ib_i}\right)\right)\left(\sum_{i=1}^n\left(\sqrt{\frac{a_i^2+b_i^2}{2}}\right)\right) \tag{2.2}$$

Lemma 3 If $0\leq x<\frac{\pi}{2}$ then the following inequalities are satisfied

$$\sin x\leq x\leq\tan x \tag{2.3}$$

Proof: Let f: $R^+\to R$ be a function defined by $f(x)=x-\sin x$, then $f'(x)=1-\cos x\geq 0$ for all $0\leq x<\frac{\pi}{2}$.We obtain $f(x)$ is an increasing function on $[0,\frac{\pi}{2})$.Then we obtain $f(x)\geq 0$. This show that: $x-\sin x\geq 0$.Also we can show $x\leq\tan x$ in the same way.

Main result:

Theorem 1. If $0\leq x<\frac{\pi}{2}$ then the following inequality is satisfied

$$2\sin x+\tan x\geq\sin x\left(1+2\left(\sqrt[4]{\frac{1+\cos^2 x}{2\cos^3 x}}\right)\right)\geq\sin x\left(1+\frac{2}{\sqrt{\cos x}}\right)\geq 3\frac{\sin x}{\sqrt[3]{\cos x}}\geq 3x$$

(3.1)

Proof: i) From (1.3) we obtain that: $\frac{\sin x}{\sqrt[3]{\cos x}}\geq x$

ii) Now we show: $\sin x\left(1+\frac{2}{\sqrt{\cos x}}\right)\geq 3\frac{\sin x}{\sqrt[3]{\cos x}}$.

When $x=0$, right equals left $(0=0)$.We assume that x ≠ 0. Then we apply Lemma 2,we obtain

$$1+\frac{2}{\sqrt{\cos x}}=1+\frac{1}{\sqrt{\cos x}}+\frac{1}{\sqrt{\cos x}}\geq 3\frac{\sin x}{\sqrt[3]{\cos x}}$$

Now if we multiply the right and left of the inequality by sin x, ii) is proved.

$$\text{iii) } 2\sin x+\tan x\geq \sin x\left(1+2\left(\sqrt[4]{\frac{1+cos^2x}{2cos^3x}}\right)\right)$$

When $x=0$, right equals left $(0=0)$.We assume that x ≠ 0. Then we apply Lemma 1,we obtain

$$2\sin x+\tan x=\sin x+\sin x+\tan x\geq \sin x+2\left(\sqrt[4]{\frac{\sin x\tan x(sin^2x+tan^2x)}{2}}\right)$$

$$=\sin x\left(1+2\left(\sqrt[4]{\frac{1+cos^2x}{2cos^3x}}\right)\right).$$ iii) is proved.

$$\text{iiii) } \sin x\left(1+2\left(\sqrt[4]{\frac{1+cos^2x}{2cos^3x}}\right)\right)\geq \sin x\left(1+\frac{2}{\sqrt{\cos x}}\right)$$

We apply Lemma 1 we get: $1+cos^2x\geq 2\cos x$.This show that iiii) is true.The proof of Theorem is complete.

Theorem 2. If $0<x<\frac{\pi}{2}$ and $s_n=\left(\frac{\sin x}{x}\right)^n+\left(\frac{\tan x}{x}\right)^{n-1}$, $n\epsilon N$ and $n\geq 2$ then the following inequality is satisfied

1) $\forall n, s_n>2$

2) $f>p, s_f>s_p$

Proof: Let's prove the theorem by mathematical induction.If $n=2$ we obtain (1.4) this show inequality is true for $n=2$.Now we show for $n=3$

If we apply Lemma 3, Lemma 1 and (1.4) respectively, we get

$$\left(\frac{\sin x}{x}\right)^3+\left(\frac{\tan x}{x}\right)^2=\frac{sin^3x+x\,tan^2x}{x^3}>\frac{sin^3x+\sin x\,tan^2x}{x^3}\geq\frac{2sin^3x\frac{1}{\cos x}}{x^3}>2.$$

Suppose the inequality is true for $n=k$.Then we must show $for\ n=k+1$.

If we apply (2.3) we obtain

$$\left(\frac{\sin x}{x}\right)^{n-1}>\left(\frac{\sin x}{x}\right)^n \text{ and } \frac{\tan x}{x}>\frac{\sin x}{x}$$

Also (1.1) show that

$$\frac{\tan x}{x}+\frac{\sin\ x}{x}-2>0$$

Then we get:

$$s_{n+1}-s_n=\left(\frac{\sin x}{x}\right)^{n+1}+\left(\frac{\tan x}{x}\right)^{n}-\left(\frac{\sin x}{x}\right)^{n}-\left(\frac{\tan x}{x}\right)^{n-1}=$$

$$=\left(\frac{\tan x}{x}\right)^{n-1}\left[\frac{\tan x}{x}-1\right]-\left(\frac{\sin x}{x}\right)^{n}\left[1-\frac{\sin\ x}{x}\right]\geq$$

$$\geq\left(\frac{\tan x}{x}\right)^{n-1}\left[\frac{\tan x}{x}-1\right]-\left(\frac{\sin x}{x}\right)^{n-1}\left[1-\frac{\sin\ x}{x}\right]\geq$$

$$\geq\left(\frac{\tan x}{x}\right)^{n-1}\left[\frac{\tan x}{x}-1\right]-\left(\frac{\tan x}{x}\right)^{n-1}\left[1-\frac{\sin\ x}{x}\right]=$$

$$=\left(\frac{\tan x}{x}\right)^{n-1}\left[\frac{\tan x}{x}+\frac{\sin\ x}{x}-2\right]>0.$$

With this we proved both 1) and 2) . The proof of Theorem is complete.

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A SIMPLE PROOF FOR MILNE'S INEQUALITY

By Daniel Sitaru-Romania

Abstract: *In this paper is presented a simple proof for Milne's inequality and a few applications.*

MILNE'S INEQUALITY $(\boldsymbol{n=2})$

If $x_1, x_2, y_1, y_2 > 0$ **then:**

$$\frac{\boldsymbol{x_1y_1}}{\boldsymbol{x_1+y_1}}+\frac{\boldsymbol{x_2y_2}}{\boldsymbol{x_2+y_2}}\le\frac{(\boldsymbol{x_1+x_2})(\boldsymbol{y_1+y_2})}{\boldsymbol{x_1+x_2+y_1+y_2}};(\boldsymbol{1})$$

Proof:

$$\frac{x_1y_1}{x_1+y_1}+\frac{x_2y_2}{x_2+y_2}=\left(x_1-\frac{x_1^2}{x_1+y_1}\right)+\left(x_2-\frac{x_2^2}{x_2+y_2}\right)=$$

$$=x_1+x_2-\left(\frac{x_1^2}{x_1+y_1}+\frac{x_2^2}{x_2+y_2}\right)\overset{Bergstrom}{\le}x_1+x_2-\frac{(x_1+x_2)^2}{x_1+y_1+x_2+y_2}=$$

$$=\frac{(x_1+x_2)^2+(x_1+x_2)(y_1+y_2)-(x_1+x_2)^2}{x_1+x_2+y_1+y_2}=\frac{(x_1+x_2)(y_1+y_2)}{x_1+x_2+y_1+y_2}$$

Equality holds for $x_1=y_1, x_2=y_2$.

MILNE'S INEQUALITY $(\boldsymbol{n=3})$

If $x_1, x_2, x_3, y_1, y_2, y_3 > 0$ **then:**

$$\frac{\boldsymbol{x_1y_1}}{\boldsymbol{x_1+y_1}}+\frac{\boldsymbol{x_2y_2}}{\boldsymbol{x_2+y_2}}+\frac{\boldsymbol{x_3y_3}}{\boldsymbol{x_3+y_3}}\le\frac{(\boldsymbol{x_1+x_2+x_3})(\boldsymbol{y_1+y_2+y_3})}{\boldsymbol{x_1+x_2+x_3+y_1+y_2+y_3}};(\boldsymbol{2})$$

Proof:

$$\frac{x_1y_1}{x_1+y_1}+\frac{x_2y_2}{x_2+y_2}+\frac{x_3y_3}{x_3+y_3}=\left(x_1-\frac{x_1^2}{x_1+y_1}\right)+\left(x_2-\frac{x_2^2}{x_2+y_2}\right)+\left(x_3-\frac{x_3^2}{x_3+y_3}\right)=$$

$$=x_1+x_2+x_3-\left(\frac{x_1^2}{x_1+y_1}+\frac{x_2^2}{x_2+y_2}+\frac{x_3^2}{x_3+y_3}\right)\overset{Bergstrom}{\leq}$$

$$\leq x_1+x_2+x_3-\frac{(x_1+x_2+x_3)^2}{x_1+y_1+x_2+y_2+x_3+y_3}=$$

$$=\frac{(x_1+x_2+x_3)^2+(x_1+x_2+x_3)(y_1+y_2+y_3)-(x_1+x_2+x_3)^2}{x_1+x_2+x_3+y_1+y_2+y_3}=$$

$$=\frac{(x_1+x_2+x_3)(y_1+y_2+y_3)}{x_1+x_2+x_3+y_1+y_2+y_3}$$

Equality holds for $x_1=y_1, x_2=y_2, x_3=y_3$.

GENERAL MILNE'S INEQUALITY: If $\boldsymbol{x_i}>0, \boldsymbol{y_i}>0, i\in\overline{\mathbf{1},\boldsymbol{n}}$ then:

$$\sum_{cyc}\frac{\boldsymbol{x_i}\boldsymbol{y_i}}{\boldsymbol{x_i}+\boldsymbol{y_i}}\leq\frac{(\sum_{i=1}^n\boldsymbol{x_i})(\sum_{i=1}^n\boldsymbol{y_i})}{\sum_{i=1}^n\boldsymbol{x_i}+\sum_{i=1}^n\boldsymbol{y_i}};\quad(\mathbf{3})$$

Proof:

$$\sum_{cyc}\frac{x_iy_i}{x_i+y_i}=\sum_{i=1}^n\left(x_i-\frac{x_i^2}{x_i+y_i}\right)=\sum_{i=1}^n x_i-\sum_{i=1}^n\frac{x_i^2}{x_i+y_i}\overset{Bergstrom}{\leq}$$

$$\leq\sum_{i=1}^n x_i-\frac{(\sum_{i=1}^n x_i)^2}{\sum_{i=1}^n x_i+\sum_{i=1}^n y_i}=\frac{(\sum_{i=1}^n x_i)^2+(\sum_{i=1}^n x_i)(\sum_{i=1}^n y_i)-(\sum_{i=1}^n x_i)^2}{\sum_{i=1}^n x_i+\sum_{i=1}^n y_i}=$$

$$=\frac{(\sum_{i=1}^n x_i)(\sum_{i=1}^n y_i)}{\sum_{i=1}^n x_i+\sum_{i=1}^n y_i}$$

Equality holds for $x_i=y_i, i\in\overline{1,n}$.

Corollary 1: If $\boldsymbol{x_1},\boldsymbol{x_2},\boldsymbol{y_1},\boldsymbol{y_2}>0, \boldsymbol{x_1}+\boldsymbol{x_2}=\boldsymbol{y_1}+\boldsymbol{y_2}$ then:

$$\frac{x_1y_1}{x_1+y_1}+\frac{x_2y_2}{x_2+y_2}\le 1$$

Proof: Replace in (1): $x_1+x_2=2, y_1+y_2=2$, we get:

$$\frac{x_1y_1}{x_1+y_1}+\frac{x_2y_2}{x_2+y_2}\le\frac{(x_1+x_2)(y_1+y_2)}{x_1+x_2+y_1+y_2}=1$$

Equality holds for $x_1=x_2=y_1=y_2=1$.

Corollary 2: If $x_1,x_2,x_3,y_1,y_2,y_3>0, x_1+x_2+x_3=y_1+y_2+y_3=3$ then:

$$\frac{x_1y_1}{x_1+y_1}+\frac{x_2y_2}{x_2+y_2}+\frac{x_3y_3}{x_3+y_3}\le\frac{3}{2}$$

Proof: Replace in (2): $x_1+x_2+x_3=y_1+y_2+y_3=3$, we get:

$$\frac{x_1y_1}{x_1+y_1}+\frac{x_2y_2}{x_2+y_2}+\frac{x_3y_3}{x_3+y_3}\le\frac{(x_1+x_2+x_3)(y_1+y_2+y_3)}{x_1+x_2+x_3+y_1+y_2+y_3}=\frac{3}{2}$$

Equality holds for $x_1=x_2=x_3=y_1=y_2=y_3=1$.

Corollary 3: If $x_i,y_i>0, i\in\overline{1,n}, x_1+x_2+\cdots+x_n=y_1+y_2+\cdots+y_n=n$, then:

$$\frac{x_1y_1}{x_1+y_1}+\frac{x_2y_2}{x_2+y_2}+\cdots+\frac{x_ny_n}{x_n+y_n}\le\frac{n}{2}, n\in\mathbb{N}, n\ge 2$$

Proof: Replace in (3): $x_1+x_2+\cdots+x_n=y_1+y_2+\cdots+y_n=n$, we get:

$$\frac{x_1y_1}{x_1+y_1}+\frac{x_2y_2}{x_2+y_2}+\cdots+\frac{x_ny_n}{x_n+y_n}\le\frac{(x_1+x_2+\cdots+x_n)(y_1+y_2+\cdots+y_n)}{x_1+x_2+\cdots+x_n+y_1+y_2+\cdots+y_n}=\frac{n\cdot n}{n+n}=\frac{n}{2}$$

Equality holds for $x_1=x_2=\cdots=x_n=y_1=y_2=\cdots=y_n=1$.

Corollary 4: If $x,y\in\mathbb{R}$ then:

$$\frac{\sin^2 x\sin^2 y}{\sin^2 x+\sin^2 y}+\frac{\cos^2 x\cos^2 y}{\cos^2 x+\cos^2 y}\le\frac{1}{2}$$

Proof: We take in (1): $x_1=\sin^2 x, x_2=\cos^2 x, x_2=\sin^2 y, y_2=\cos^2 y$, we get:

$$\frac{\sin^2 x \sin^2 y}{\sin^2 x+\sin^2 y}+\frac{\cos^2 x \cos^2 y}{\cos^2 x+\cos^2 y}\le \frac{(\sin^2 x+\cos^2 x)(\sin^2 y+\cos^2 y)}{\sin^2 x+\cos^2 x+\sin^2 y+\cos^2 y}=\frac{1}{2}$$

Equality holds for $x=y=\frac{\pi}{4}$.

Application 1: If a,b,c,r,R are sides, inradii and circumradii in ΔABC, a',b',c',r',R' are sides, inradii and circumradii in $\Delta A'B'C'$ then:

$$\frac{aa'}{a+a'}+\frac{bb'}{b+b'}+\frac{cc'}{c+c'}\le \frac{3\sqrt{3}RR'}{2(r+R')}$$

Solution. In (2) we take: $x_1=a, y_1=a', x_2=b, y_2=b', x_3=c, y_3=c', 2s=a+b+c$,

$2s'=a'+b'+c'$, hence, we have:

$$\frac{aa'}{a+a'}+\frac{bb'}{b+b'}+\frac{cc'}{c+c'}\le \frac{(a+b+c)(a'+b'+c')}{(a+b+c)+(a'+b'+c')}=\frac{2s\cdot 2s'}{2s+2s'}=$$

$$=\frac{2ss'}{s+s'}\overset{Mitrinovic}{\le}\frac{2\cdot\frac{3\sqrt{3}}{2}R\cdot\frac{3\sqrt{3}}{2}R'}{3\sqrt{3}r+3\sqrt{3}r'}=\frac{R\cdot\frac{3\sqrt{3}}{2}R'}{r+r'}=\frac{3\sqrt{3}RR'}{2(r+r')}$$

Equality holds for $a=b=c$ and $a'=b'=c'$.

Application 2: In acute triangles $ABC, A'B'C'$ the following relationship holds:

$$\frac{1}{\cot A+\cot A'}+\frac{1}{\cot B+\cot B'}+\frac{1}{\cot C+\cot C'}\le \frac{1}{\cot A\cot B\cot C+\cot A'\cot B'\cot C'}$$

Solution:

$$\sum_{cyc}\frac{1}{\cot A+\cot A'}=\sum_{cyc}\frac{1}{\frac{1}{\tan A}+\frac{1}{\tan A'}}=\sum_{cyc}\frac{\tan A\tan A'}{\tan A+\tan A'}=$$

$$=\sum_{cyc}\left(\tan A-\frac{\tan^2 A}{\tan A+\tan A'}\right)=\sum_{cyc}\tan A-\sum_{cyc}\frac{\tan^2 A}{\tan A+\tan A'}\overset{Bergstrom}{\le}$$

$$\le\sum_{cyc}\tan A-\frac{(\sum\tan A)^2}{\sum\tan A+\sum\tan A'}=$$

$$=\frac{(\sum\tan A)^2+\sum\tan A\cdot\sum\tan A'-(\sum\tan A')^2}{\sum\tan A+\sum\tan A'}=$$

$$=\frac{\prod\tan A\cdot\prod\tan A'}{\prod\tan A+\prod\tan A'}=\frac{1}{\prod\frac{1}{\tan A}+\prod\frac{1}{\tan A'}}=\frac{1}{\prod\cot A+\prod\cot A'}$$

Equality holds for $A=B=C=A'=B'=C'=\frac{\pi}{3}$

Reference: ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

A SIMPLE PROOF FOR SANDHAM'S INEQUALITY

By Daniel Sitaru-Romania

SANDHAM'S INEQUALITY: If $p,q\in\mathbb{N}\setminus\{0\}$ then:

$$H_p+H_q\le 1+H_{pq},\qquad H_n=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n},\qquad n\in\mathbb{N}^*$$

Proof: $-1+H_q=-1+\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{q}\right)=$

$$=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{q}=\frac{p}{2p}+\frac{p}{3p}+\frac{p}{4p}+\cdots+\frac{p}{pq}\le$$

$$\le\left(\frac{1}{p+1}+\frac{1}{p+2}+\cdots+\frac{1}{2p}\right)+\left(\frac{1}{2p+1}+\frac{1}{2p+2}+\cdots+\frac{1}{3p}\right)+\cdots$$

$$\ldots+\left(\frac{1}{(q-1)p+1}+\frac{1}{(q-1)p+2}+\cdots+\frac{1}{pq}\right)=H_{pq}-H_p$$

$$-1+H_q\le H_{pq}-H_p,\ \ H_p+H_q\le 1+H_{pq}\quad (1)$$

Equality holds for $p=q=1$

Corollary 1: If $p,q,r\in\mathbb{N}\setminus\{0\}$ then:

$$H_p+H_q+H_r\le 2+H_{pqr}$$

Proof: $H_p + H_q + H_r \overset{(1)}{\leq} 1 + H_{pq} + H_r \overset{(1)}{\leq} 1 + 1 + H_{(pq)r} = 2 + H_{pqr}$

Corollary 2: If $m, n, p, q \in \mathbb{N} \setminus \{0\}$ then:

$$\boldsymbol{H_m + H_n + H_p + H_q \leq 3 + H_{mnpq}}$$

Proof: $H_m + H_n + H_p + H_q \overset{(1)}{\leq} 1 + H_{mn} + 1 + H_{pq} =$

$$= 2 + H_{mn} + H_{pq} \overset{(1)}{\leq} 2 + 1 + H_{(mn)(pq)} = 3 + H_{mnpq}$$

Equality holds for $m = n = p = 1 = 1$.

Reference: [1] Romanian Mathematical Magazine – www.ssmrmh.ro

INEQUALITIES WITH GOLDEN RATIO USING KEPLER'S TRIANGLE

By Daniel Sitaru – Romania

Abstract. *In this article we will use Kepler's triangle for proving inequalities involving golden ratio.*

MAIN RESULTS:

If φ is golden ratio $\left(\varphi = \frac{1+\sqrt{5}}{2}\right)$ then:

$$\frac{\left(1+\sqrt{\varphi}+\varphi\right)^2}{4} + \frac{5\varphi}{\left(1+\sqrt{\varphi}+\varphi\right)^2} > \frac{8\varphi}{1+\sqrt{\varphi}+\varphi}; \quad (1)$$

$$\frac{\left(1+\sqrt{\varphi}+\varphi\right)^2}{4} < \varphi^2 + \frac{2\varphi\sqrt{\varphi}}{1+\sqrt{\varphi}+\varphi} + \frac{3\varphi}{\left(1+\sqrt{\varphi}+\varphi\right)^2}; \quad (2)$$

$$2\varphi + \frac{\varphi}{1+\sqrt{\varphi}+\varphi} > \frac{\sqrt{3}\left(1+\sqrt{\varphi}+\varphi\right)}{2}; \quad (3)$$

$$\frac{\varphi\sqrt{\varphi}+1}{\sqrt{\varphi}\left(1+\sqrt{\varphi}\right)} + \frac{\sqrt{\varphi}}{1+\varphi} + \frac{2}{\sqrt{\varphi}\left(1+\sqrt{\varphi}+\varphi\right)} < 2; \quad (4)$$

$$\frac{\boldsymbol{\varphi^2\sqrt{\varphi}+1}}{\boldsymbol{\sqrt{\varphi}(1+\sqrt{\varphi})}}+\frac{\boldsymbol{\varphi}}{\boldsymbol{\varphi+1}}<\frac{\boldsymbol{3\sqrt{3}\varphi}}{\boldsymbol{16}}\sqrt{\boldsymbol{\varphi}-\frac{\boldsymbol{2\sqrt{\varphi}}}{\boldsymbol{1+\sqrt{\varphi}+\varphi}}};\ (\mathbf{5})$$

$$\sqrt[3]{\frac{\boldsymbol{\varphi}}{\boldsymbol{1+\sqrt{\varphi}-\varphi}}}+\sqrt[3]{\frac{\boldsymbol{\sqrt{\varphi}}}{\boldsymbol{\varphi+\sqrt{\varphi}-1}}}+\sqrt[3]{\frac{\mathbf{1}}{\boldsymbol{1+\varphi-\sqrt{\varphi}}}}<\frac{\boldsymbol{3\sqrt{\varphi}}}{\boldsymbol{4(1+\sqrt{\varphi}+\varphi)}};\ (\mathbf{6})$$

KEPLER'S TRIANGLE: Let ΔABC be a triangle with sides

$$a=\varphi; b=1; c=\sqrt{\varphi}; \varphi=\frac{1+\sqrt{5}}{2}$$

It is clear that: $1+\varphi>\sqrt{\varphi}; 1+\sqrt{\varphi}>\varphi;\ \varphi+\sqrt{\varphi}>1$

$$\varphi^2=\left(\frac{1+\sqrt{5}}{2}\right)^2=\frac{1+5+2\sqrt{5}}{4}=1+\frac{1+\sqrt{5}}{2}=1+\varphi$$

$$\varphi^2=1+\varphi\Rightarrow a^2=b^2+c^2.$$

This right triangle is called Kepler's triangle.

$$F=\frac{bc}{2}=\frac{\sqrt{\varphi}}{2}(\boldsymbol{area})$$

$$s=\frac{a+b+c}{2}=\frac{\varphi+1+\sqrt{\varphi}}{2}(\boldsymbol{semiperimeter})$$

$$\sin B=\frac{1}{\varphi}=\cos C; \sin C=\frac{1}{\sqrt{\varphi}}=\cos B$$

$$r=\frac{F}{s}=\frac{\sqrt{\varphi}}{1+\sqrt{\varphi}+\varphi}(\boldsymbol{inradii})$$

$$m_a=\frac{a}{2}=\frac{\varphi}{2}$$

$$m_b^2=\frac{1}{2}(a^2+c^2)-\frac{b^2}{4}=\frac{1}{2}(b^2+2c^2)-\frac{b^2}{4}=\frac{b^2}{4}+c^2=\frac{1}{4}+\varphi=\frac{1+4\varphi}{4}$$

$$m_b=\frac{\sqrt{1+4\varphi}}{2}$$

$$m_c^2=\frac{1}{2}(a^2+b^2)-\frac{c^2}{4}=\frac{1}{2}(2b^2+c^2)-\frac{c^2}{4}=b^2+\frac{c^2}{4}=1+\frac{\varphi}{4}=\frac{4+\varphi}{4}$$

$$m_c=\frac{\sqrt{4+\varphi}}{2}$$

Medians: $m_a=\frac{\varphi}{4}; m_b=\frac{\sqrt{1+4\varphi}}{2}; m_c=\frac{\sqrt{4+\varphi}}{2}$

$$R=\frac{a}{2}=m_a=\frac{\varphi}{2}(\boldsymbol{circumradii})$$

$$h_a=\frac{bc}{a}=\frac{1\cdot\sqrt{\varphi}}{\varphi}=\frac{1}{\sqrt{\varphi}}$$

$$h_c=c=\sqrt{\varphi}; h_b=b=1$$

Altitudes: $h_a=\frac{1}{\sqrt{\varphi}}; h_b=1; h_c=\sqrt{\varphi}$

Proof for main result:

1) We will prove that (1);(2) are equivalent with **Gerretsen's** inequality:

$$\frac{\left(1+\sqrt{\varphi}+\varphi\right)^2}{4}+\frac{5\varphi}{\left(1+\sqrt{\varphi}+\varphi\right)^2}>\frac{8\varphi\sqrt{\varphi}}{1+\sqrt{\varphi}+\varphi}$$

$$\frac{\left(1+\sqrt{\varphi}+\varphi\right)^2}{4}>16\cdot\frac{\varphi}{2}\cdot\frac{\sqrt{\varphi}}{1+\sqrt{\varphi}+\varphi}-5\cdot\frac{\varphi}{\left(1+\sqrt{\varphi}+\varphi\right)^2}$$

$$s^2\geq 16Rr-5r^2 \text{ (Gerretsen)}$$

2. $\frac{\left(1+\sqrt{\varphi}+\varphi\right)^2}{4}<\varphi^2+\frac{2\varphi\sqrt{\varphi}}{1+\sqrt{\varphi}+\varphi}+\frac{3\varphi}{\left(1+\sqrt{\varphi}+\varphi\right)^2}$

$$\frac{\left(1+\sqrt{\varphi}+\varphi\right)^2}{4}<4\cdot\frac{\varphi^2}{4}+4\cdot\frac{\varphi}{2}\cdot\frac{\sqrt{\varphi}}{1+\sqrt{\varphi}+\varphi}+\frac{3\varphi}{\left(1+\sqrt{\varphi}+\varphi\right)^2}$$

$$s^2\leq 4R^2+4Rr+3r^3 \text{(Gerretsen)}$$

3. We will prove that (3) is equivalent with **Doucet's** inequality:

$$2\varphi+\frac{\sqrt{\varphi}}{1+\sqrt{\varphi}+\varphi}>\frac{\sqrt{3}\left(1+\sqrt{\varphi}+\varphi\right)}{2}$$

$$4\cdot\frac{\varphi}{2}+\frac{\sqrt{\varphi}}{1+\sqrt{\varphi}+\varphi}>\frac{\sqrt{3}\left(1+\sqrt{\varphi}+\varphi\right)}{2}$$

$$4R+r\geq s\sqrt{3}\text{(Doucet)}$$

4. We will prove that (4) is equivalent with problem **4212-Crux Mathematicorum**

$$\frac{\varphi\sqrt{\varphi}+1}{\sqrt{\varphi}\left(1+\sqrt{\varphi}\right)}+\frac{\sqrt{\varphi}}{1+\varphi}+\frac{2}{\sqrt{\varphi}\left(1+\sqrt{\varphi}+\varphi\right)}<2$$

$$\frac{\varphi}{1+\sqrt{\varphi}}+\frac{1}{\sqrt{\varphi}\left(1+\sqrt{\varphi}\right)}+\frac{\sqrt{\varphi}}{1+\sqrt{\varphi}}+\frac{2}{\sqrt{\varphi}\left(1+\sqrt{\varphi}+\varphi\right)}<2$$

$$\frac{\varphi}{1+\sqrt{\varphi}}+\frac{1}{\sqrt{\varphi}+\varphi}+\frac{\sqrt{\varphi}}{1+\varphi}+\frac{\frac{\sqrt{\varphi}}{1+\sqrt{\varphi}+\varphi}}{\frac{\varphi}{2}}<2$$

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{r}{R}\leq 2$$

5. We will prove that (5) is equivalent with problem **4462-Crux Mathematicorum**

$$\frac{\varphi^2\sqrt{\varphi}+1}{\sqrt{\varphi}\left(1+\sqrt{\varphi}\right)}+\frac{\varphi}{\varphi+1}<\frac{3\sqrt{3}\varphi}{16}\sqrt{\varphi-\frac{2\sqrt{\varphi}}{1+\sqrt{\varphi}+\varphi}}$$

$$\frac{\varphi^2\sqrt{\varphi}+1}{\sqrt{\varphi}\left(1+\sqrt{\varphi}\right)}+\frac{\varphi}{\varphi+1}<\frac{3\sqrt{3}\varphi}{16}\sqrt{\frac{\varphi^2+\varphi+\varphi\sqrt{\varphi}-2\sqrt{\varphi}}{1+\sqrt{\varphi}+\varphi}}$$

$$\frac{\varphi^2\sqrt{\varphi}+1}{\sqrt{\varphi}\left(1+\sqrt{\varphi}\right)}+\frac{\varphi}{\varphi+1}<\frac{3\sqrt{6}\varphi}{16}\sqrt{\frac{\varphi+\varphi\sqrt{\varphi}+\varphi^2-2\sqrt{\varphi}}{2\left(1+\sqrt{\varphi}-\varphi\right)}}$$

$$\frac{\varphi^2}{1+\sqrt{\varphi}}+\frac{1}{\sqrt{\varphi}\left(1+\sqrt{\varphi}\right)}+\frac{\varphi}{\varphi+1}<\frac{3\sqrt{6}}{4}\cdot\frac{\varphi}{2}\cdot\frac{1+\sqrt{\varphi}+\varphi}{\sqrt{\varphi}}\cdot\frac{\sqrt{\varphi}}{2}\sqrt{\frac{\varphi}{2}-\frac{\sqrt{\varphi}}{1+\sqrt{\varphi}+\varphi}}$$

$$\frac{\varphi^2}{1+\sqrt{\varphi}}+\frac{1}{\sqrt{\varphi}+\varphi}+\frac{\varphi}{\varphi+1}<\frac{3\sqrt{6}}{4}\cdot\frac{\frac{\varphi}{2}}{\frac{\sqrt{\varphi}}{1+\sqrt{\varphi}+\varphi}}\cdot\sqrt{\frac{\varphi}{2}\left(\frac{\varphi}{2}-\frac{\sqrt{2}}{1+\sqrt{\varphi}+\varphi}\right)}$$

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\le\frac{3\sqrt{6}R}{4r}\sqrt{R(R-r)}$$

6. We will prove that (6) is equivalent to problem **4289-Mathematicorum**

$$\sqrt[3]{\frac{\varphi}{1+\sqrt{\varphi}-\varphi}}+\sqrt[3]{\frac{\sqrt{\varphi}}{\varphi+\sqrt{\varphi}-1}}+\sqrt[3]{\frac{1}{1+\varphi-\sqrt{\varphi}}}<\frac{3\sqrt{\varphi}}{4\left(1+\sqrt{\varphi}+\varphi\right)}$$

$$\sqrt[3]{\frac{\frac{\sqrt{\varphi}}{1+\sqrt{\varphi}+\varphi}}{\frac{1}{\sqrt{\varphi}}}}+\sqrt[3]{\frac{\frac{\sqrt{\varphi}}{\sqrt{\varphi}+\varphi-1}}{1}}+\sqrt[3]{\frac{\frac{\sqrt{\varphi}}{1+\varphi-\sqrt{\varphi}}}{\sqrt{\varphi}}}<\frac{3}{2}\cdot\frac{\frac{\varphi}{2}}{\frac{\sqrt{\varphi}}{1+\sqrt{\varphi}+\varphi}}$$

$$\sqrt[3]{\frac{r_a}{h_a}}+\sqrt[3]{\frac{r_b}{h_b}}+\sqrt[3]{\frac{r_c}{h_c}}\le\frac{3R}{2r}$$

Conclusions: Any metric relationship in triangle generating a relationship (identity or inequality) with golden ratio. The key is to identify in a given relationship with golden ratio the elements of Kepler's triangle.

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FAMOUS INEQUALITIES IN TRIANGLE REDESIGNED

WITH CONWAY' METHOD

By Daniel Sitaru-Romania

Abstract: *In this paper is presented the Conway's method used for redesign famous inequalities in triangle as Euler, Mitrinovic, Leibniz, Ionescu-Weitzenbock, Leuenberger, Steining, Băndilă, Neuberg, Gordon, Goldner, Curry, Doucet, Gotman and Minkowski.*

MAIN RESULT:

If $x, y, z \in \mathbb{R}$ **are such that:** $x+y>0, y+z>0, z+x>0, xy+yz+zx>0$ **then:**

$$1)\ \sum_{cyc}(x+y)\sqrt{(x+z)(y+z)} \ge 4(xy+yz+zx)$$

$$2)\ \left(\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}\right)^2 \ge 6\sqrt{3(xy+yz+zx)}$$

$$3)\ 2\left(\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}\right)\sqrt{xy+yz+zx} \le 3\sqrt{3(x+y)(y+z)(z+x)}$$

$$4)\ 8(x+y+z)(xy+yz+zx) \le 9(x+y)(y+z)(z+x)$$

$$5)\ x+y+z \ge \sqrt{3(xy+yz+zx)}$$

$$6)\ \frac{1}{\sqrt{x+y}}+\frac{1}{\sqrt{y+z}}+\frac{1}{\sqrt{z+x}} \ge 2\sqrt{\frac{3(xy+yz+zx)}{(x+y)(y+z)(z+x)}}$$

$$7)\ \frac{1}{\sqrt{x+y}}+\frac{1}{\sqrt{y+z}}+\frac{1}{\sqrt{z+x}} \le \frac{\sqrt{3}\left(\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}\right)}{2\sqrt{xy+yz+zx}}$$

$$8)\ \frac{1}{\sqrt{(x+y)(x+z)}}+\frac{1}{\sqrt{(y+z)(y+x)}}+\frac{1}{\sqrt{(z+x)(z+y)}} \ge \frac{4(xy+yz+zx)}{(x+y)(y+z)(z+x)}$$

$$9)\ \frac{1}{\sqrt{(x+y)(x+z)}}+\frac{1}{\sqrt{(y+z)(y+x)}}+\frac{1}{\sqrt{(z+x)(z+y)}} \le \frac{\left(\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}\right)^2}{\sqrt{(x+y)(y+z)(z+x)}}$$

10) $\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}\geq\frac{\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}}{\sqrt{(x+y)(y+z)(z+x)}}$

11) $\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}\leq\frac{\left(\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}\right)^2}{4(xy+yz+zx)}$

12) $\sqrt{\frac{x+y}{x+z}}+\sqrt{\frac{x+z}{x+y}}\leq\frac{\left(\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}\right)\sqrt{(x+y)(y+z)(z+x)}}{2(xy+yz+zx)}$

13) $(x+y+z)\left(\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}\right)^2\geq 18(xy+yz+zx)$

14) $\sqrt{(x+y)(x+z)}+\sqrt{(y+z)(y+x)}+\sqrt{(z+x)(z+y)}\geq\frac{36(xy+yz+zx)}{\left(\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}\right)^2}$

15) $\sqrt{(x+y)(x+z)}+\sqrt{(y+z)(y+x)}+\sqrt{(z+x)(z+y)}\leq\frac{9(x+y)(y+z)(z+x)}{4(xy+yz+zx)}$

16) $\sqrt{(x+y)(x+z)}+\sqrt{(y+z)(y+x)}+\sqrt{(z+x)(z+y)}\geq 2\sqrt{3(xy+yz+zx)}$

17) $x^2+y^2+z^2\geq xy+yz+zx$

18) $(x+y)(x+z)+(y+z)(y+x)+(z+x)(z+y)\geq 4(xy+yz+zx)$

19) $\frac{9\sqrt{(x+y)(y+z)(z+x)}}{\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}}\geq 2\sqrt{3(xy+yz+zx)}$

20) $\sqrt{3}\left(\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}\right)\leq 4\sqrt{\frac{(x+y)(y+z)(z+x)}{xy+yz+zx}}+\frac{2\sqrt{xy+yz+zx}}{\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}}$

21) $\sqrt{x+\frac{y+z}{4}}+\sqrt{y+\frac{z+x}{4}}+\sqrt{z+\frac{x+y}{4}}\geq\frac{9\sqrt{xy+yz+Zx}}{\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}}$

22) $\sqrt{x+\frac{y+z}{4}}+\sqrt{y+\frac{z+x}{4}}+\sqrt{z+\frac{x+y}{4}}\leq\frac{9\sqrt{(x+y)(y+z)(z+x)}}{4\sqrt{xy+yz+zx}}$

23) $\frac{3\sqrt{xy+yz+zx}}{\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}}\leq\sqrt[4]{\frac{3}{4}(xy+yz+zx)}$

CONWAY'S METHOD

Let be $x, y, z \in \mathbb{R}$ such that $x + y > 0, y + z > 0, z + x > 0, xy + yz + Zx > 0$. Denote $a = \sqrt{x + y}, b = \sqrt{y + z}, c = \sqrt{z + x}$. We will prove that a, b, c can be sides in a triangle ABC: $a + b > c \Leftrightarrow \sqrt{x + y} + \sqrt{y + z} > \sqrt{z + x}$

$$x + y + y + z + 2\sqrt{(x + y)(y + z)} > z + x$$

$$y + \sqrt{(x + y)(y + z)} > 0, \qquad y + \sqrt{xy + yz + zx + y^2} > y + |y| \geq 0$$

Analogous: $b + c > a, c + a > b$. The semiperimeter $s = \frac{1}{2}\left(\sqrt{x + y} + \sqrt{y + z} + \sqrt{z + x}\right)$.

We will prove that: $F = \frac{1}{2}\sqrt{xy + yz + zx}$.

$$s - a = \frac{1}{2}\left(\sqrt{x + y} + \sqrt{y + z} - \sqrt{z + x}\right), \qquad s - b = \frac{1}{2}\left(\sqrt{x + y} + \sqrt{y + z} - \sqrt{z + x}\right)$$

$$s - c = \frac{1}{2}\left(\sqrt{y + z} + \sqrt{z + x} - \sqrt{x + y}\right)$$

$$s(s - a) = \frac{1}{4}\left(\left(\sqrt{x + y}\right)^2 + \left(\sqrt{z + x}\right)^2 - \left(\sqrt{y + z}\right)^2\right) =$$

$$= \frac{1}{4}\left(x + y + z + x + 2\sqrt{(x + y)(z + x)} - y - z\right) =$$

$$= \frac{1}{4}\left(2\sqrt{(x + y)(z + x)} + 2x\right) = \frac{1}{2}\left(\sqrt{(x + y)(z + x)} + x\right)$$

$$(s - b)(s - c) = \frac{1}{4}\left(\left(\sqrt{y + z}\right)^2 - \left(\sqrt{x + y} - \sqrt{z + x}\right)^2\right) =$$

$$= \frac{1}{4}\left(y + z - x - y - z - x + 2\sqrt{(x + y)(z + x)}\right) = \frac{1}{2}\left(\sqrt{(x +)(z + x)} - x\right)$$

$$F^2 = s(s - a)(s - b)(s - c) = \frac{1}{4}\left(\left(\sqrt{(x + y)(z + x)}\right)^2 - x^2\right) =$$

$$= \frac{1}{4}(x^2 + xy + yz + zx - x^2) = \frac{1}{4}(xy + yz + zx)$$

$$\boldsymbol{F=\frac{1}{2}\sqrt{xy+yz+zx}}$$

Let R, r be inradii and circumradii in ΔABC:

$$\boldsymbol{r=\frac{F}{s}=\frac{\sqrt{xy+yz+zx}}{\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}}}$$

$$\boldsymbol{R=\frac{abc}{4F}=\frac{\sqrt{(x+y)(y+z)(z+x)}}{2\sqrt{xy+yz+zx}}}$$

Let m_a, m_b, m_c be medians in ΔABC:

$$m_a^2=\frac{1}{2}(b^2+c^2)-\frac{1}{4}a^2=\frac{1}{2}(x+z+x+y)-\frac{1}{4}(y+z)=$$

$$=\frac{2(2x+y+z)-y-z}{4}=\frac{4x+y+z}{4}$$

$$\boldsymbol{m_a^2=\frac{4x+y+z}{4}}$$

Analogous:

$$\boldsymbol{m_b^2=\frac{x+4y+z}{4}}\text{ and }\boldsymbol{m_c^2=\frac{x+y+4z}{4}}$$

Let h_a, h_b, h_c be altitudes in ΔABC:

$$\boldsymbol{h_a=\frac{2F}{a}=\sqrt{\frac{xy+yz+zx}{y+z}}=\sqrt{x+\frac{yz}{y+z}}}$$

Analogous:

$$\boldsymbol{h_b=\sqrt{\frac{xy+yz+zx}{z+x}}=\sqrt{y+\frac{zx}{z+x}}}$$

$$\boldsymbol{h_c=\sqrt{\frac{xy+yz+zx}{x+y}}=\sqrt{z+\frac{xy}{x+y}}}$$

Proof of 1): In ΔABC the following relationship holds:

$$\boldsymbol{R \geq 2r\ (Euler)}$$

Replace the values of R, r with Conway's substitutions:

$$\frac{\sqrt{(x+y)(y+z)(z+x)}}{2\sqrt{xy+yz+zx}} \geq \frac{2\sqrt{xy+yz+zx}}{\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}}$$

$$\boldsymbol{\sum_{cyc}(x+y)\sqrt{(z+x)(z+y)} \geq 4(xy+yz+zx)}$$

Proof of 2): In ΔABC the following relationship holds:

$$\boldsymbol{s \geq 3\sqrt{3}r\ (Mitrinovic)}$$

Replace the values s, r with Conway's substitutions:

$$\frac{\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}}{2} \geq \frac{3\sqrt{3}\sqrt{xy+yz+zx}}{\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}}$$

$$\boldsymbol{\left(\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}\right)^2 \geq 6\sqrt{3(xy+yz+zx)}}$$

Proof of 3): In ΔABC the following relationship holds:

$$\boldsymbol{s \leq \frac{3\sqrt{3}}{2}R\ (Mitrinovic)}$$

Replace the values of s, R with Conway's substitutions:

$$\frac{\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}}{2} \leq \frac{3\sqrt{3}}{2}\cdot\frac{\sqrt{(x+y)(y+z)(z+x)}}{2\sqrt{xy+yz+zx}}$$

$$\boldsymbol{2\left(\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}\right)\sqrt{xy+yz+zx} \leq 3\sqrt{3(x+y)(y+z)(z+x)}}$$

Proof of 4): In ΔABC the following relationship holds:

$$\boldsymbol{a^2+b^2+c^2 \leq 9R^2\ (Leibniz)}$$

Replace the values of a, b, c, R with Conway's substitutions:

$$\left(\sqrt{x+y}\right)^2+\left(\sqrt{y+z}\right)^2+\left(\sqrt{z+X}\right)^2\le 9\cdot\frac{(x+y)(y+z)(z+x)}{4(xy+yz+zx)}$$

$$2(x+y+z)\le\frac{9(x+y)(y+z)(z+x)}{4(xy+yz+zx)}$$

$$8(x+y+z)(xy+yz+zx)\le 9(x+y)(y+z)(z+x)$$

$$\boldsymbol{8(x+y+z)(xy+yz+zx)\le 9(x+y)(y+z)(z+x)}$$

Proof of 5): In ΔABC the following relationship holds:

$$\boldsymbol{a^2+b^2+c^2\ge 4\sqrt{3}F\ (Ionescu-Weitzenbock)}$$

Replace the values of a,b,c,F with Conway's substitutions:

$$\left(\sqrt{x+y}\right)^2+\left(\sqrt{y+z}\right)^2+\left(\sqrt{z+x}\right)^2\ge 4\sqrt{3}\cdot\frac{1}{2}\sqrt{xy+yz+zx}$$

$$2(x+y+z)\ge 2\sqrt{3(xy+yz+zx)}$$

$$\boldsymbol{x+y+z\ge\sqrt{3(xy+yz+zx)}}$$

Proof of 6): In ΔABC the following relationship holds:

$$\boldsymbol{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge\frac{\sqrt{3}}{R}\ (Leuenberger)}$$

Replace the values of a,b,c,R with Conway's substitutions:

$$\frac{1}{\sqrt{x+y}}+\frac{1}{\sqrt{y+z}}+\frac{1}{\sqrt{z+x}}\ge\sqrt{3}\cdot\frac{2\sqrt{xy+yz+zx}}{\sqrt{(x+y)(y+z)(z+x)}}$$

$$\boldsymbol{\frac{1}{\sqrt{x+y}}+\frac{1}{\sqrt{y+z}}+\frac{1}{\sqrt{z+x}}\ge 2\sqrt{\frac{3(xy+yz+zx)}{(x+y)(y+z)(z+x)}}}$$

Proof of 7): In ΔABC the following relationship holds:

$$\boldsymbol{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\le\frac{\sqrt{3}}{2r}\ (Leuenberger)}$$

Replace the values of a, b, c, r with Conway's substitutions:

$$\frac{1}{\sqrt{x+y}}+\frac{1}{\sqrt{y+z}}+\frac{1}{\sqrt{z+x}}\le\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}}{\sqrt{xy+yz+zx}}$$

Proof of 8): In ΔABC the following relationship holds:

$$\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\ge\frac{1}{R^2}\ \ (Leuenberger)$$

Replace the values of a, b, c, R with Conway's substitutions:

$$\frac{1}{\sqrt{(x+y)(x+z)}}+\frac{1}{\sqrt{(y+z)(y+x)}}+\frac{1}{\sqrt{(z+x)(z+y)}}\ge\frac{4(xy+yz+zx)}{(x+y)(y+z)(z+x)}$$

Proof of 9): In ΔABC the following relationship holds:

$$\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\le\frac{1}{4r^2}\ \ (Leuenberger)$$

Replace the values of a, b, c, r with Conway's substitutions:

$$\frac{1}{\sqrt{(x+y)(x+z)}}+\frac{1}{\sqrt{(y+z)(y+x)}}+\frac{1}{\sqrt{(z+x)(z+y)}}\le\frac{\left(\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}\right)^2}{4(xy+yz+zx)}$$

Proof of 10): In ΔABC the following relationship holds:

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\ge\frac{1}{2Rr}\ \ (Steining)$$

Replace the values of a, b, c, r, R with Conway's substitutions:

$$\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}\ge\frac{\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}}{\sqrt{(x+y)(y+z)(z+x)}}$$

Proof of 11): In ΔABC the following relationship holds:

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\le\frac{1}{4r^2}\ \ (Steining)$$

Replace the values of a, b, c, r, R with Conway's substitutions:

$$\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}\le\frac{\left(\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}\right)^2}{4(xy+yz+zx)}$$

Proof of 12): In ΔABC the following relationship holds:

$$\frac{b}{c}+\frac{c}{b}\le\frac{R}{r}\ (Băndilă)$$

Replace the values of a,b,c,r,R with Conway's substitutions:

$$\sqrt{\frac{x+y}{x+z}}+\sqrt{\frac{x+z}{x+y}}\le\frac{\left(\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}\right)\sqrt{(x+y)(y+z)(z+x)}}{2(xy+yz+zx)}$$

Proof of 13): In ΔABC the following relationship holds:

$$a^2+b^2+c^2\ge 36r^2\ (Neuberg)$$

Replace the values of a,b,c,r with Conway's substitutions:

$$(x+y+z)\left(\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}\right)^2\ge 18(xy+yz+zx)$$

Proof of 14): In ΔABC the following relationship holds:

$$ab+bc+ca\ge 36r^2\ (Leuenberger)$$

Replace the values of a,b,c,r with Conway's substitutions:

$$\sqrt{(x+y)(x+z)}+\sqrt{(y+z)(y+x)}+\sqrt{(z+x)(z+y)}\ge\frac{36(xy+yz+zx)}{\left(\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}\right)^2}$$

Proof of 15): In ΔABC the following relationship holds:

$$ab+bc+ca\le 9R^2\ (Leuenberger)$$

Replace the values of a,b,c,R with Conway's substitutions:

$$\sqrt{(x+y)(x+z)}+\sqrt{(y+z)(y+x)}+\sqrt{(z+x)(z+y)}\le\frac{9(x+y)(y+z)(z+x)}{4(xy+yz+zx)}$$

Proof of 16): In ΔABC the following relationship holds:

$$ab+bc+ca\ge 4\sqrt{3}F\ (Gordon)$$

Replace the values of a,b,c,F with Conway's substitutions:

$$\sqrt{(x+y)(x+z)}+\sqrt{(y+z)(y+x)}+\sqrt{(z+x)(z+y)}\ge 2\sqrt{3(xy+yz+zx)}$$

Proof of 17): In ΔABC the following relationship holds:

$$a^4+b^4+c^4\ge 16F^2\ (Goldner)$$

Replace the values of a,b,c,F with Conway's substitutions:

$$\left(\sqrt{x+y}\right)^4+\left(\sqrt{y+z}\right)^4+\left(\sqrt{z+x}\right)^4\ge 16\cdot\frac{1}{4}(xy+yz+zx)$$

$$(x+y)^2+(y+z)^2+(z+x)^2\ge 4(xy+yz+zx)$$

Proof of 18): In ΔABC the following relationship holds:

$$a^2b^2+b^2c^2+c^2a^2\ge 16F^2\ (Goldner)$$

Replace the values of a,b,c,F with Conway's substitutions:

$$(x+y)(x+z)+(y+z)(y+x)+(z+x)(z+y)\ge 4(xy+yz+zx)$$

Proof of 19): In ΔABC the following relationship holds:

$$\frac{9abc}{a+b+c}\ge 4\sqrt{3}F\ (Curry)$$

Replace the values of a,b,c,F with Conway's substitutions:

$$\frac{9\sqrt{(x+y)(y+z)(z+x)}}{\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}}\ge 2\sqrt{3(xy+yz+zx)}$$

Proof of 20): In ΔABC he following relationship holds:

$$s\sqrt{3}\le 4R+r\ (Doucet)$$

Replace the values of s,R,r with Conway's substitutions:

$$\sqrt{3}\left(\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}\right)$$
$$\leq 4\sqrt{\frac{(x+y)(y+z)(z+x)}{xy+yz+zx}}+\frac{2\sqrt{xy+yz+zx}}{\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}}$$

Proof of 21): In ΔABC the following relationship holds:

$$m_a+m_b+m_c\geq 9r\ (Gotman)$$

Replace the values of m_a, m_b, m_c with Conway's substitutions:

$$\sqrt{x+\frac{y+z}{4}}+\sqrt{y+\frac{z+x}{4}}+\sqrt{z+\frac{x+y}{4}}\geq\frac{9\sqrt{xy+yz+zx}}{\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}}$$

Proof of 22): In ΔABC the following relationship holds:

$$m_a+m_b+m_c\leq\frac{9R}{2}\ (Gotman)$$

Replace the values of m_a, m_b, m_c with Conway's substitutions:

$$\sqrt{x+\frac{y+z}{4}}+\sqrt{y+\frac{z+x}{4}}+\sqrt{z+\frac{x+y}{4}}\leq\frac{9\sqrt{(x+y)(y+z)(z+x)}}{4\sqrt{xy+yz+zx}}$$

Proof of 23): In ΔABC the following relationship holds:

$$\frac{3}{\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}}\leq\sqrt[4]{3F^2}\ (Makowski)$$

Replace the values of h_a, h_b, h_c, F with Conway's substitutions:

$$\frac{3\sqrt{xy+yz+zx}}{\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}}\leq\sqrt[4]{\frac{3}{4}(xy+yz+zx)}$$

OBSERVATIONS: Recall these famous inequalities in triangles: a, b, c –sides, r, R –inradii, circumradii, F –area, h_a, h_b, h_c –altitudes, m_a, m_b, m_c –medians, s –semiperimeter.

$$1)\ R\geq 2r\ (Euler)$$

$$2)\ s \geq 3\sqrt{3}r\ (Mirinovic - I)$$

$$3)\ s \leq \frac{3\sqrt{3}}{2}R\ (Mitrinovic - II)$$

$$4)\ a^2 + b^2 + c^2 \leq 9R^2\ (Leibniz)$$

$$5)\ a^2 + b^2 + c^2 \geq 4\sqrt{3}F\ (Ionescu - Weitzenbock)$$

$$6)\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{\sqrt{3}}{R}\ (Leuenberger - I)$$

$$7)\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{\sqrt{3}}{2r}\ (Leuenberger - II)$$

$$8)\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \geq \frac{1}{R^2}\ (Leuenberger - III)$$

$$9)\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \leq \frac{1}{4r^2}\ (Leuenberger - IV)$$

$$10)\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq \frac{1}{2Rr}\ (Steining - I)$$

$$11)\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \leq \frac{1}{4r^2}\ (Steining - II)$$

$$12)\frac{b}{c} + \frac{c}{b} \leq \frac{R}{r}(\ Bandila)$$

$$13)\ a^2 + b^2 + c^2 \geq 36r^2\ (Neuberg)$$

$$14)\ ab + bc + ca \geq 36r^2\ (Leuenberger - V)$$

$$15)\ ab + bc + ca \leq 9R^2\ (Leuenberger - VI)$$

$$16)\ ab + bc + ca \geq 4\sqrt{3}\ (Gordon)$$

$$17)\ a^4 + b^4 + c^4 \geq 16F^2 (Goldner - I)$$

$$18)\ a^2b^2 + b^2c^2 + c^2a^2 \geq 16F^2\ (Goldner - II)$$

$$19)\ \frac{9abc}{a+b+c} \geq 4F\sqrt{3}\ (Curry)$$

$$20)\ s\sqrt{3} \leq 4R + r\ (Doucet)$$

$$21)\ m_a + m_b + m_c \geq 9r\ (Gotman - I)$$

$$22)\ m_a + m_b + mc \leq \frac{9R}{2}\ (Gotman - II)$$

$$23)\ \frac{3}{\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}} \leq \sqrt[4]{3F^2}\ (Makowski)$$

REFERENCE: ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

PROPOSED PROBLEMS

PROBLEMS FOR JUNIORS

J.2101 If $x, y, z > 0$, then in ΔABC the following relationship holds:

$$\frac{x \cdot a\sqrt{a}}{(y+z)\sqrt{h_a}} + \frac{y \cdot b\sqrt{b}}{(z+x)\sqrt{h_b}} + \frac{z \cdot c\sqrt{c}}{(x+y)\sqrt{h_c}} \geq \sqrt{6F}$$

Proposed by D.M.Bătinețu-Giurgiu, Claudia Nănuți-Romania

J.2102 If $x, y, z > 0$, then in ΔABC the following relationship holds:

$$\frac{x\sqrt{a}}{(y+z)\sqrt{h_a}} + \frac{y\sqrt{b}}{(z+x)\sqrt{h_b}} + \frac{z\sqrt{c}}{(x+y)\sqrt{h_c}} \geq \frac{1}{2} \cdot \sqrt[4]{108}$$

Proposed by D.M.Bătinețu-Giurgiu,Claudia Nănuți-Romania

J.2103 In ΔABC the following relationship holds: $a^2(s-a)+b^2(s-b)+c^2(s-c)\ge 4\sqrt[4]{3}\cdot\sqrt{F^3}$

Proposed by D.M.Bătinețu-Giurgiu-Claudia Nănuți-Romania

J.2104 In ΔABC the following relationship holds:

$$\frac{a^3}{b+c}+\frac{b^3}{c+a}+\frac{c^3}{a+b}\ge 2\sqrt{3}\cdot F$$

Proposed by D.M.Bătinețu-Giurgiu,Dan Nănuți-Romania

J.2105 In $\Delta ABC, G$ −centroid and $d_a=d(G,BC), d_b=d(G,CA), d_c=d(G,AB)$ holds:

$$\frac{a^4+b^4}{d_ad_b}+\frac{b^4+c^4}{d_bd_c}+\frac{c^4+a^4}{d_cd_a}\ge 96\sqrt{3}\cdot F$$

Proposed by D.M.Bătinețu-Giurgiu, Dan Nănuți-Romania

J.2106 In $\Delta ABC, G$ −centroid and $d_a=d(G,BC), d_b=d(G,CA), d_c=d(G,AB)$ holds:

$$\frac{x^2a^4+y^2b^4}{d_bd_c}+\frac{x^2b^4+y^2c^4}{d_cd_a}+\frac{x^2c^4+y^2a^4}{d_ad_b}\ge 24\sqrt{3}(x+y)^2\cdot F;\forall x,y>0$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

J.2107 In ΔABC the following relationship holds:

$$\frac{a^4+b^4}{h_ah_b}+\frac{b^4+c^4}{h_bh_c}+\frac{c^4+a^4}{h_ch_a}\ge\frac{32}{\sqrt{3}}\cdot F$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

J.2108 If $M\in Int(\Delta ABC), d_a=d(M,BC), d_b=d(M,CA), d_c=d(M,AB)$, then:

$$\frac{a}{d_a}+\frac{b}{d_b}+\frac{c}{d_c}\ge 6\sqrt{3}$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

J.2109 If $x,y\ge 0, x+y>0$, then in ΔABC the following relationship holds:

$$\frac{x^2a^4+y^2b^4}{m_bm_c}+\frac{x^2b^4+y^2c^4}{m_cm_a}+\frac{x^2c^4+y^2a^4}{m_am_b}\ge\frac{8(x+y)^2}{\sqrt{3}}\cdot F$$

Proposed by D.M.Bătinețu-Giurgiu,Claudia Nănuți-Romania

J.2110 If $M\in Int(\Delta ABC), x=AM, y=BM, z=CM$, then:

$$\frac{x^{2022}h_a^{86}}{a^{1936}}+\frac{y^{2022}h_b^{86}}{b^{1936}}+\frac{z^{2022}h_c^{86}}{c^{1936}}\ge\frac{2^{86}\cdot F^{86}}{3^{2021}}$$

Proposed by D.M.Bătinețu-Giurgiu,Claudia Nănuți-Romania

J.2111 If $x, y, z > 0$, then in ΔABC the following relationship holds:

$$\frac{(y+z)a}{xh_a} + \frac{(z+x)b}{yh_b} + \frac{(x+y)c}{zh_c} \geq 4\sqrt{3}$$

Proposed by D.M.Bătinețu-Giurgiu,Dan Nănuți-Romania

J.2112 If $x, y, z > 0$, then in ΔABC the following relationship holds:

$$\frac{a^4 e^{x^2}}{y+z} + \frac{b^4 e^{y^2}}{z+x} + \frac{c^4 e^{z^2}}{x+y} > 16F^2$$

Proposed by D.M.Bătinețu-Giurgiu,Dan Nănuți-Romania

J.2113 In ΔABC the following relationship holds:

$$\frac{xa}{(y+z)h_a} + \frac{yb}{(z+x)h_b} + \frac{zc}{(x+y)h_c} \geq \sqrt{3}; \forall x, y, z > 0$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

J.2114 In ΔABC the following relationship holds:

$$\frac{a^2}{(b+\sqrt{bc}+\sqrt{ca})h_a} + \frac{b^2}{(c+\sqrt{ca}+\sqrt{ab})h_b} + \frac{c^2}{(a+\sqrt{ab}+\sqrt{bc})h_c} \geq \frac{2\sqrt{3}}{3}$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

J.2115 If $x, y, z > 0$ then in ΔABC the following relationship holds:

$$\frac{ae^x}{(y+z+2)h_a} + \frac{be^y}{(z+x+2)h_b} + \frac{ce^z}{(x+y+2)h_c} \geq \sqrt{3}$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

J.2116 If $x, y, z \geq 0$, then in ΔABC the following relationship holds:

$$\frac{a^2 e^x}{y+z+2} + \frac{b^2 e^y}{z+x+2} + \frac{c^2 e^z}{x+y+2} \geq 2\sqrt{3} \cdot F$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

J.2117 If $x, y, z > 0$, then in ΔABC the following relationship holds:

$$\frac{e^x}{(y+z+2)h_a^2} + \frac{e^y}{(z+x+2)h_b^2} + \frac{e^z}{(x+y+2)h_c^2} \geq \frac{\sqrt{3}}{2F}$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

J.2118 If $m, n > 0, x, y, z \geq 0$ then in ΔABC the following relationship holds:

$$\frac{a^4e^x}{my+nz+m+n}+\frac{b^4e^y}{mz+nx+m+n}+\frac{c^4e^z}{mx+ny+m+n}\ge\frac{16}{m+n}\cdot F^2$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

J.2119 In ΔABC the following relationship holds:

$$\frac{a^3}{b+\sqrt[3]{ac^2}}+\frac{b^3}{c+\sqrt[3]{ba^2}}+\frac{c^3}{a+\sqrt[3]{cb^2}}\ge 2\sqrt{3}\cdot F$$

Proposed by D.M.Bătinețu-Giurgiu, Gheorghe Boroica-Romania

J.2120 If $x,y,z\ge 0$, then in ΔABC the following relationship holds:

$$\frac{(e^x+e^y)a}{(z+1)h_b}+\frac{(e^y+e^z)b}{(x+1)h_c}+\frac{(e^z+e^x)c}{(y+1)h_a}\ge 4\sqrt{3}$$

Proposed by D.M.Bătinețu-Giurgiu, Gheorghe Boroica-Romania

J.2121 Let ΔABC and ΔUVW with length sides $u=\sqrt{a}, v=\sqrt{b}, w=\sqrt{c}$ and area Δ, then:

$$\Delta\ge\frac{\sqrt[4]{3}}{2}\cdot\sqrt{F}$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

J.2122 In ΔABC the following relationship holds:

$$s\ge\sqrt[4]{27}\cdot\sqrt{F}+\frac{1}{4}\cdot\sum_{cyc}\left(\sqrt{a}-\sqrt{b}\right)^2$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

J.2123 Let $t,u,v\ge 0, x,y,z>0$ and triangles $A_iB_iC_i, i=\overline{1,3}$ with circumradius $R_i, i=\overline{1,3}$, then:

$$\frac{x+y}{za_1^ta_2^ua_3^v}+\frac{y+z}{xb_1^tb_2^ub_3^v}+\frac{z+x}{yc_1^tc_2^uc_3^v}\ge\frac{6}{\left(\sqrt{3}\right)^{t+u+v}R_1^tR_2^uR_3^v}$$

Proposed by D.M.Bătinețu-Giurgiu, Claudia Nănuți-Romania

J.2124 If $x,y>0$ then in ΔABC the following relationship holds:

$$(ax+by)^4h_ah_b+(bx+cy)^4h_bh_c+(cx+ay)^4h_ch_a\ge 256\sqrt{3}\cdot x^2y^2F^3$$

Proposed by D.M.Bătinețu-Giurgiu, Claudia Nănuți-Romania

J.2125 Let $x,y,z>0$ and w_a internal bisector, then in ΔABC holds:

$$\frac{x\sqrt{w_b w_c}}{y+z}\cdot\frac{a^4}{\sqrt{h_b h_c}}+\frac{y\sqrt{w_c w_a}}{z+x}\cdot\frac{b^4}{\sqrt{h_c h_a}}+\frac{z\sqrt{w_a w_b}}{x+y}\cdot\frac{c^4}{\sqrt{h_a h_b}}\geq 8F^2$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

J.2126 In ΔABC the following relationship holds:

$$\frac{1}{h_a h_b}+\frac{1}{h_b h_c}+\frac{1}{h_c h_a}\geq\frac{\sqrt{3}}{F}$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

J.2127 If $x,y,z>0$ then in ΔABC the following relationship holds:

$$\frac{x^2a^3}{(y+z)^2h_a}+\frac{y^2b^3}{(z+x)^2h_b}+\frac{z^2c^3}{(x+y)^2h_c}\geq 2F$$

Proposed by D.M.Bătinețu-Giurgiu, Nicolae Mușuroia-Romania

J.2128 Let $m,n,t\geq 0, x,y,z>0$ such that $m+n=2t+2$, then in ΔABC holds:

$$\frac{x^{t+1}a^m}{(y+z)^{t+1}h_a^n}+\frac{y^{t+1}b^m}{(z+x)^{t+1}h_b^n}+\frac{z^{t+1}c^m}{(x+y)^{t+1}h_c^n}\geq 2\left(\sqrt{3}\right)^{1-t}\left(\sqrt{F}\right)^{m-n}$$

Proposed by D.M.Bătinețu-Giurgiu,Nicolae Mușuroia-Romania

J.2129 If $x,y>0$ and $A_1B_1C_1, A_2B_2C_2$ are triangles with altitudes h_{a_1},h_{b_1},h_{c_1} and h_{a_2},h_{b_2},h_{c_2} respectively, then holds:

$$\frac{1}{h_{a_1}^x h_{a_2}^y}+\frac{1}{h_{b_1}^x h_{b_2}^y}+\frac{1}{h_{c_1}^x h_{c_2}^y}\geq\frac{\left(\sqrt{3}\right)^{2-x-y}}{\left(\sqrt{F_1}\right)^x\left(\sqrt{F_2}\right)^y}$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

J.2130 In ΔABC the following relationship holds:

$$\frac{a}{\left(b+\sqrt{bc}+\sqrt{ca}\right)h_a^2}+\frac{b}{\left(c+\sqrt{ca}+\sqrt{ab}\right)h_b^2}+\frac{c}{\left(a+\sqrt{ab}+\sqrt{bc}\right)h_c^2}\geq\frac{\sqrt{3}}{3F}$$

Proposed by D.M.Bătinețu-Giurgiu, Claudia Nănuți-Romania

J.2131 In ΔABC the following relationship holds:

$$\frac{a^3}{b+\sqrt{bc}+\sqrt[3]{abc}}+\frac{b^3}{c+\sqrt{ca}+\sqrt[3]{abc}}+\frac{c^3}{a+\sqrt{ab}+\sqrt[3]{abc}}\geq\frac{4\sqrt{3}}{3}\cdot F$$

Proposed by D.M.Bătinețu-Giurgiu, Claudia Nănuți-Romania

J.2132 Let $m, n, t \geq 0, x, y, z > 0$ such that $m + n = 4t + 4$, then in ΔABC holds:

$$\frac{x^{t+1}a^m}{(y+z)^{t+1}h_a^n} + \frac{y^{t+1}b^m}{(z+x)^{t+1}h_b^n} + \frac{z^{t+1}c^m}{(x+y)^{t+1}h_c^n} \geq \frac{2^{3t-n-3}}{3^t} \cdot F^{2t-n+2}$$

Proposed by D.M.Bătinețu-Giurgiu,Dan Nănuți-Romania

J.2133 If $x, y > 0$ then in ΔABC the following relationship holds:

$$\frac{a^3}{bx + y\sqrt{ca}} + \frac{b^3}{cx + y\sqrt{ab}} + \frac{c^3}{ax + y\sqrt{bc}} \geq \frac{4\sqrt{3}}{x+y} \cdot F$$

Proposed by D.M.Bătinețu-Giurgiu,Dan Nănuți-Romania

J.2134 If $m \geq 0, x, y > 0$ then in ΔABC the following relationship holds:

$$\frac{a^{m+2}}{\left(xb + y\sqrt{ca}\right)^m} + \frac{b^{m+2}}{\left(xc + y\sqrt{ab}\right)^m} + \frac{c^{m+2}}{\left(xa + y\sqrt{bc}\right)^m} \geq \frac{4\sqrt{3}}{(x+y)^m} \cdot F$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

J.2135 Solve for integers: $a^2 + 2b = \frac{a^2}{2b}$

Proposed by Soumava Chakraborty-India

J.2136 In $\Delta ABC, v_a$ —Bevan's cevian, the following relationship holds:

$$\sum_{cyc} v_a \geq 6R - 2R\sum_{cyc}\left(\frac{a}{b+c}\right)^2 \geq \sum_{cyc} h_a$$

Proposed by Soumava Chakraborty-India

J.2137 In $\Delta ABC, p_a$ —Spieker's cevian, the following relationship:

$$\sum_{cyc} \frac{(p_a^2 - m_a^2)(2s+a)^2}{8s^2 - a^2} \geq 2r(R - 2r)$$

Proposed by Soumava Chakraborty- India

J.2138 In $\Delta ABC, p_a$ —Spieker's cevian, the following relationship holds:

$$\sum_{cyc} \frac{p_a^2(2s+a)^2}{b^2 - bc + c^2} \leq 16s^2$$

Proposed by Soumava Chakraborty- India

J.2139 Solve for natural numbers: $2a^2b^2 = 3a^2 + 4b$

Proposed by Soumava Chakraborty- India

J.2140 In $\Delta ABC, p_a$ −Spieker's cevian, the following relationship holds:

$$\sum_{cyc}(p_a^2 - m_a^2)(2s+a)^2 \le 2R(R-2r)(11s^2+4Rr+r^2)$$

Proposed by Soumava Chakraborty- India

J.2141 Solve for real numbers:

$$\begin{cases} a^2 - b + ab = 1 \\ abc + a - 2b = 0 \\ 2b^2 - a^2 + bc = 2 \end{cases}$$

Proposed by Soumava Chakraborty- India

J.2142 In ΔABC the following relationship holds:

$$\sum_{cyc} a^2(n_a+g_a)^4 - 16\sum_{cyc} a^2 m_a^4 + 4s\sum_{cyc}(b-c)^4(s-a) \ge 64F^2(s^2-12Rr-3r^2)$$

Proposed by Soumava Chakraborty- India

J.2143 In ΔABC the following relationship holds:

$$\sum_{cyc} m_a m_b \ge \frac{3\left(2-\sqrt{3}\right)}{8}\sum_{cyc} a^2 + \frac{9rs}{2}$$

Proposed by Soumava Chakraborty- India

J.2144 In $\Delta ABC, p_a$ −Spieker's cevian, the following relationship holds:

$$\sum_{cyc}\frac{p_a}{\cos\frac{A}{2}} \ge 2s$$

Proposed by Soumava Chakraborty- India

J.2145 In $\Delta ABC, p_a$ −Spieker's cevian, the following relationship holds:

$$\sum_{cyc}\frac{p_a w_a r_a}{(s-b)(s-c)} \ge \frac{s^2}{r}$$

Proposed by Soumava Chakraborty- India

J.2146 Solve for integers:

$$\begin{cases} 2a^2 + 4b^2 = a^2b^2 - 2 \\ 2ab + 2ac + 2c + c^2 = 0 \end{cases}$$

Proposed by Soumava Chakraborty- India

J.2147 Let $a,b,c \geq 0, a+b+c=3$. Prove that $a^2+b^2+c^2+\sqrt{ab}+\sqrt{bc}+\sqrt{ca} \geq 0$.

Proposed by Phan Ngoc Chau-Vietnam

J.2148 Let $a,b,c > 0, a+b+c=2$ and $m=\min\{2ab^2+b^3c, 2bc^2+c^3a, 2ca^2+a^3b\}$. Solve the following equation $x^2+2x+m=0$.

Proposed by Phan Ngoc Chau-Vietnam

J.2149 Let $a,b,c,d > 0$. Prove that:

$$\sqrt{\frac{1}{a(1+bc)}+\frac{1}{b(1+cd)}}+\sqrt{\frac{1}{c(1+da)}+\frac{1}{d(1+ab)}} \geq \frac{2\sqrt{2}}{\sqrt{1+abcd}}$$

Proposed by Tran Quoc Thinh-Vietnam

J.2150 Let $a,b,c > 0, a+b+c=1$. Prove that:

$$\frac{a(a^2-bc)}{abc(a^2-bc)+3(bc)^2}+\frac{b(b^2-ca)}{abc(b^2-ca)+3(ca)^2}+\frac{c(c^2-ab)}{abc(c^2-ab)+3(ab)^2} \geq 0$$

Proposed by Phan Ngoc Chau-Vietnam

J.2151 Solve for real numbers:

$$\sqrt{(1-x)(2-x)}+\sqrt{(2-x)(3-x)}+\sqrt{(3-x)(1-x)}=1+x$$

Proposed by Tran Quoc Thinh-Vietnam

J.2152 Let $a,b,c,d > 0$. Prove that:

$$\frac{1}{a(1+bc)}+\frac{1}{b(1+cd)}+\frac{1}{c(1+da)}+\frac{1}{d(1+ab)} \geq \frac{4}{1+abcd}$$

Proposed by Tran Quoc Thinh-Vietnam

J.2153 If $a,b,c,d > 0$ and $a^3+b^3+c^3+d^3=1$ then:

$$\frac{1}{2} \geq \frac{a^2b^2}{a+b}+\frac{b^2c^2}{b+c}+\frac{c^2d^2}{c+d}+\frac{d^2a^2}{d+a}$$

Proposed by Alex Szoros-Romania

J.2154 If $x,y,z \geq 0$ and $x+y+z=1$ then prove:

$$\frac{yz}{x+1}+\frac{zx}{y+1}+\frac{xy}{z+1} \leq \frac{1}{4}$$

Proposed by Hikmat Mammadov-Azerbaijan

J.2155 If $x,y,z > 0$ and $\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}=1$ then prove that:

$$(x-1)(y-1)(z-1) \leq 6\sqrt{3}-10$$

Proposed by Hikmat Mammadov-Azerbaijan

J.2156 Solve the system:

$$\begin{cases} p^2 - 12q = 15 \\ \dfrac{p^2}{8q} + \dfrac{2q}{3} = \sqrt{\dfrac{p^3}{3q} + \dfrac{p^2}{4} - \dfrac{q}{2}} \end{cases}$$

Proposed by Hikmat Mammadov-Azerbaijan

J.2157 Find: $\min\limits_{-1\leq x\leq 1}\left(\sqrt{74-70x}+\sqrt{130-126\sqrt{1-x^2}}\right)$

Proposed by Neculai Stanciu-Romania

J.2158 In ΔABC the following relationship holds:

$$\frac{ar_a}{r_b+r_c}+\frac{br_b}{r_c+r_a}+\frac{cr_c}{r_a+r_b} \geq \frac{F}{r}$$

Proposed by Ertan Yildirim- Turkiye

J.2159 In ΔABC the following relationship holds:

$$\left[2\left(\cos^2\frac{B}{7}+\cos^2\frac{C}{7}\right)-\cos^2\frac{A}{7}\right]\prod_{cyc}\left(\cos\frac{B}{7}+\cos\frac{C}{7}-\cos\frac{A}{7}\right) \leq \cos^2\frac{B}{7}\cos^2\frac{C}{7}\left(\cos\frac{A}{7}+\cos\frac{B}{7}+\cos\frac{C}{7}\right)$$

Proposed by Bogdan Fuștei-Romania

J.2160 If P is a point in plane of ΔABC, then the following relationship holds:

$$AP\cdot\cos\frac{A}{2}+BP\cdot\cos\frac{B}{2}+CP\cdot\cos\frac{C}{2} \geq s$$

Proposed by Bogdan Fuștei-Romania

J.2161 If $M \in \mathbb{R}^3, x, y, z > 0$ then in ΔABC the following relationship holds:

$$\frac{MA^2}{x}+\frac{MB^2}{y}+\frac{MC^2}{z} \geq xyz\left(\frac{a+b+c}{xy+yz+zx}\right)^2$$

Proposed by Bogdan Fuștei-Romania

J.2162 In ΔABC the following relationship holds:

$$\sum_{cyc} bg_a(r_b+r_c) \geq (4R+r)\sqrt{\sum_{cyc} a^2b^2}$$

Proposed by Bogdan Fuștei-Romania

J.2163 In ΔABC the following relationship holds:

$$m_b + 2w_a \le \frac{\sqrt{3}}{2}(2c+b)$$

Proposed by Bogdan Fuștei-Romania

J.2164 In ΔABC the following relationship holds:

$$\sqrt{\frac{R}{2r}} \ge \frac{s\sqrt{3}}{\sqrt{(\sum h_a)^2 - \frac{3}{4}(\max\{h_a,h_b,h_c\} - \min\{h_a,h_b,h_c\})^2}}$$

Proposed by Bogdan Fuștei-Romania

J.2165 Let $\omega_1 = \sqrt{a\sqrt{n_a} - b\sqrt{n_b} + x\sqrt{n_c}}$ and $\omega_2 = \sqrt{a\sqrt{n_a} + b\sqrt{n_b} - c\sqrt{n_c}}$. In ΔABC holds: $\frac{(b\sqrt{n_b} - c\sqrt{n_c})^2}{\sum a\sqrt{n_a}} \le a\sqrt{n_a} - \omega_1 \cdot \omega_2$

Proposed by Bogdan Fuștei-Romania

J.2166 In $\Delta ABC, \omega_1 = \max\left\{\frac{m_a}{h_a}, \frac{m_b}{h_b}, \frac{m_c}{h_c}\right\}$ and $\omega_2 = \min\left\{\frac{m_a m_b}{h_a h_b}, \frac{m_b m_c}{h_b h_c}, \frac{m_c m_a}{h_c h_a}\right\}$ then holds:

$$\frac{R}{r} \ge \frac{2}{3}\left(\omega_1 + \omega_2 + \sqrt{\prod_{cyc} \frac{m_a}{h_a}}\right)$$

Proposed by Bogdan Fuștei-Romania

J.2167 In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{m_a}{h_a} \cdot \sum_{cyc} \frac{h_a}{m_a} \le \frac{9(R+2r)^2}{8Rr}$$

Proposed by Bogdan Fuștei-Romania

J.2168 In ΔABC the following relationship holds:

$$n_a^2 + n_b^2 + n_c^2 + 2s^2 = 3(g_a^2 + g_b^2 + g_c^2) + 4(R-2r)(h_a + h_b + h_c)$$

Proposed by Bogdan Fuștei-Romania

J.2169 In ΔABC the following relationship holds:

$$1 + \frac{R}{r} \ge \frac{m_a}{m_b} + \frac{m_b}{m_c} + \frac{m_c}{m_a}$$

Proposed by Bogdan Fuștei-Romania

J.2170 Solve for integers: $z^3 = 2241 + 3(x+y)(z-x)(z-y)$

Proposed by Jamal Issah-Somalia

J.2171 O −circumcenter, I −incenter in ΔABC. Prove that:

$$\sum_{cyc} \frac{\cos(\widehat{OAI})}{a^2 + 9R^2} \le \frac{1}{16r^2}$$

Proposed by Gheorghe Molea-Romania

J.2172 If $a, b, c > 0, a^2 + b^2 + c^2 = 1$ then:

$$\sum_{cyc} ab(3a^2b + 4ab - 2c^2 + 1) \le 2$$

Proposed by Gheorghe Molea-Romania

J.2173 In ΔABC the following relationship holds: $\sum_{cyc} r_a^2 \left(\frac{1}{2m_a + m_b} + \frac{1}{2m_a + m_c}\right) \ge \frac{12r^2}{R}$

Proposed by Gheorghe Molea-Romania

J.2174 If $x, y > 0$ such that $x - \sqrt{x+2} = \sqrt{y+3} - y$, then determine $\min\{x+y\}$ and $\max\{x+y\}$.

Proposed by Neculai Stanciu-Romania

J.2175 If $x^3 - x + 3 = 0$ has the roots a, b and c then determine the monic polynomial with the roots a^5, b^5 and c^5.

Proposed by Neculai Stanciu-Romania

J.2176 Determine all pairs (x, y) of integers which satisfy $|x^2 - y^2| - \sqrt{16y + 1} = 0$.

Proposed by Neculai Stanciu-Romania

J.2177 Determine all positive integers $a, b, c, d, x, y, z, t, a \ne b \ne c \ne d$ which satisfy

$$a + b + c = td, b + c + d = xa, c + d + a = yb, d + a + b = zc.$$

Proposed by Neculai Stanciu-Romania

J.2178 If $a, b, c > 0, abc = 1$ then:

$$\frac{1+a}{2+b} + \frac{1+b}{2+c} + \frac{1+c}{2+a} \le \frac{2(a+b+c)}{3}$$

Proposed by Tran Quoc Thinh-Vietnam

J.2179 In ΔABC the following relationship holds:

$$2\sum_{cyc} \frac{1}{h_a}\sqrt{m_b m_c (r_a + r_c - h_b)(r_a + r_b - h_c)} \ge \sum_{cyc} \frac{b+c}{a}$$

Proposed by Bogdan Fuștei-Romania

J.2180 If in $\Delta ABC, \omega$ −Brocard's angle then:

$$\sin\omega \leq 4 \cdot \sqrt[3]{\prod_{cyc} \frac{m_a^2}{4a^2+b^2+c^2}}$$

Proposed by Bogdan Fuștei-Romania

J.2181 In $\Delta ABC, n_a$ −Nagel's cevian holds:

$$\sum_{cyc}\left(\frac{2n_a h_b}{n_b h_a} - \frac{n_a h_c}{n_c h_a}\right) \geq 3$$

Proposed by Bogdan Fuștei-Romania

J.2182 Solve for real numbers: $\sqrt[7]{32}x^{14} - 7x^8 + 6\sqrt[7]{32} = 0$

Proposed by Carlos Paiva-Brazil

J.2183 Solve for real numbers:

$$x^7 - 14x^6 + 77x^5 - 210x^4 + 294x^3 - 196x^2 + 49x - 16 = 0$$

Proposed by Carlos Paiva-Brazil

J.2184 Let $x, y, z > 0, x^2 + y^2 + z^2 = 2(xy + yz + zx)$. Find the minimum value of

$$P = \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y}$$

Proposed by Hung Nguyen Viet-Vietnam

J.2185 Solve for real numbers:

$$\begin{cases} \left(\frac{x}{y}\right)^{\frac{y}{x}} = \frac{x^y}{y^x} \\ \sqrt[2019]{\frac{\frac{x}{y}+\frac{y}{x}}{x^y+y^x}} = e^{\frac{\frac{x}{y}-\frac{y}{x}}{x^x \cdot y^y}} \end{cases}$$

Proposed by Urfan Aliyev-Azerbaijan

J.2186 If $a, b, c > 0, a^2 + b^2 + c^2 + ab + bc + ca = 6$ then:

$$\frac{1}{(a+b)^2} + \frac{1}{(b+c)^2} + \frac{1}{(c+a)^2} \geq \frac{3}{4}$$

Proposed by Hoang Le Nhat Tung-Vietnam

J.2187 If $a, b, c > 0, abc(a+b+c)^3 = 27$ then: $(a+b)(b+c)(c+a) \geq 8$

Proposed by Hoang Le Nhat Tung-Vietnam

J.2188 In ΔABC the following relationship holds:

$$\frac{m_b h_c}{a}+\frac{m_c h_a}{b}+\frac{m_a h_b}{c}\le\frac{9\sqrt{3}}{4}R$$

Proposed by Marian Ursărescu-Romania

J.2189 Find the minimum value of $\lambda\ge 0$ so that the inequality is true in any triangle ABC:

$$\frac{3R}{2r}\ge\sum_{cyc}\frac{(a+b)^4}{a^4+\lambda a^2b^2+b^4}$$

Proposed by Alex Szoros-Romania

J.2190 Let $a,b,c,d>0$ such that $\frac{1}{1+a^2}+\frac{1}{1+b^2}+\frac{1}{1+c^2}+\frac{1}{1+d^2}=1$ then:

$$\sqrt{a+b}+\sqrt{c+d}\ge\sqrt{3}\left(\sqrt{\frac{1}{a}+\frac{1}{b}}+\sqrt{\frac{1}{c}+\frac{1}{d}}\right)$$

Proposed by Tran Quoc Thinh-Vietnam

J.2191 O −circumcenter, P −Brocard's point in $\Delta ABC, O_a, O_b, O_c$ −circumcenters of $\Delta BPC, \Delta CPA, \Delta APB$. Prove that:

$$\frac{\sqrt{3}}{4}\frac{R^2}{r^2}\ge\sum_{cyc}\frac{OO_a}{a}\ge\sqrt{3}$$

Proposed by Eldeniz Hesenov-Georgia

J.2192 If $a,b,c>0$ such that $a^2+b^2+c^2=1$ then prove:

$$\sum_{cyc}ab(3a^3b+4ab-2c^2+1)\le 2$$

Proposed by Gheorghe Molea-Romania

J.2193 In acute ΔABC holds:

$$\sqrt[2021]{\tan^n A}+\sqrt[2021]{\tan^n B}+\sqrt[2021]{\tan^n C}\ge\frac{n\left(3\sqrt{3}-1\right)+6063}{2021}$$

Proposed by Phan Ngoc Chau-Vietnam

J.2194 Let $a,b,c\ge 0, a+b+c=1$. Prove that: $\sqrt{a+1}+\sqrt{2b+1}+\sqrt{3c+1}\ge 2+\sqrt{2}$

Proposed by Phan Ngoc Chau-Vietnam

J.2195 Let $a,b,c>0$. Prove that:

$$\sqrt{\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{b}{a}+\frac{a}{c}+\frac{c}{b}\right)} \geq 1+\sqrt[3]{\left(1+\frac{bc}{a^2}\right)\left(1+\frac{ca}{b^2}\right)\left(1+\frac{ab}{c^2}\right)}$$

Proposed by Phan Ngoc Chau-Vietnam

J.2196 Let $a, b, c > 0$ then:

$$\sqrt[4]{\frac{c^3(a+b)}{2}}+\sqrt[4]{\frac{a^3(b+c)}{2}}+\sqrt[4]{\frac{b^3(c+a)}{2}} \geq \sqrt{ab}+\sqrt{bc}+\sqrt{ca}$$

Proposed by Phan Ngoc Chau-Vietnam

J.2197 If $x, y \in \mathbb{R}$, then in ΔABC the following relationship holds:

$$\frac{x^2m_a^8+y^2m_b^8}{c^4}+\frac{x^2m_b^8+y^2m_c^8}{a^4}+\frac{x^2m_c^8+y^2m_a^8}{b^4} \geq \frac{81(x+y)^2}{32}F^2$$

Proposed by Alex Szoros-Romania

J.2198 Let ΔABC and $\Delta A_1B_1C_1$ with sides a, b, c and a_1, b_1, c_1, respectively, F, F_1 −area, $P \in Int(\Delta ABC), d_a, d_b, d_c$ −distances from P to the sides a, b, c. Prove that:

$$a_1BP \cdot CP + b_1AP \cdot CP + c_1AP \cdot BP \geq \sqrt{\frac{1}{2}\sum_{cyc}(aAP)^2(b_1^2+c_1^2-a_1^2)+\frac{4FF_1d_ad_bd_c}{R}}$$

Proposed by Bogdan Fuștei-Romania

J.2199 $a, b, c, r, R, a', b', c', r', R'$ −sides, inradii, circumradii in ΔABC and $\Delta A'B'C'$. If $a' = \sqrt{a}, b' = \sqrt{b}, c' = \sqrt{c}$, then:

$$\frac{72(r')^3}{R} \leq \sum_{cyc(a,b,c)}\frac{h_a}{h'_a} \leq \frac{9(R')^3}{R}$$

Proposed by Mehmet Șahin-Turkiye

J.2200 Let $a_1, a_2, \dots, a_{2021} \in \mathbb{N}^*$ such that $a_1^3+a_2^3+\cdots+a_{2021}^3 \leq 16161$. Prove that:

$$a_1+a_2+\cdots+a_{2021} \leq 4041$$

Proposed by Phan Ngoc Chau-Vietnam

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

PROBLEMS FOR SENIORS

S.2101 If $x, y \geq 0, x + y > 0, A_1A_2 \ldots A_n, n \geq 3$ convex polygon with sides $a_k = A_kA_{k+1}, \forall k = \overline{1,n}$ and $B_k = \frac{a_k^3}{xa_k + ya_{k+1}} + \frac{a_{k+1}^3}{xa_{k+1} + ya_{k+2}} + \frac{a_{k+2}^3}{xa_{k+2} + ya_k}, k = \overline{1,n}$,

$$a_{k+2} = a_2, a_{k+3} = a_3, \text{ then: } \sum_{k=1}^{n} B_k \geq \frac{12F}{x+y} \cdot \tan\frac{\pi}{n}$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

S.2102 Let $f: (0, \infty) \to (0, \infty), f(x) = \frac{x^4+1}{\sqrt{x^4 - x^2 + 1}}$, then in ΔABC holds:

$$\frac{f(x)a^3}{(y+z)h_a} + \frac{f(v)}{(z+x)h_b} + \frac{f(w)}{(x+y)h_c} \geq 8F; x, y, z \in (0, \infty)$$

Proposed by D.M.Bătinețu-Giurgiu, Claudia Nănuți-Romania

S.2103 Let $f: (0, \infty) \to (0, \infty), f(x) = \frac{x^4+1}{\sqrt{x^4 - x^2 + 1}}$, then:

$$\frac{f(x)}{v+w} + \frac{f(v)}{w+u} + \frac{f(w)}{u+v} \geq 3; \forall u, v, w > 0$$

Proposed by D.M.Bătinețu-Giurgiu, Claudia Nănuți-Romania

S.2104 Let $f: \mathbb{R}_+^* \to (0, \infty), f(t) = \frac{t^4+1}{\sqrt{t^4 - t^2 + 1}}$ then in ΔABC holds:

$$f(x+y)ab + f(y+z)bc + f(z+x)ca \geq 8(xy + yz + zx)F; \forall x, y, z > 0$$

Proposed by D.M.Bătinețu-Giurgiu,Dan Nănuți-Romania

S.2105 If $f: \mathbb{R} \to \mathbb{R}, 1 + f(x+y) \leq f(x) + f(y) \leq x + y + 2, \forall x, y \in \mathbb{R}$, then in ΔABC holds:

$$\frac{f(x^2)}{y+z} \cdot a^2 + \frac{f(y^2)}{z+x} \cdot b^2 + \frac{f(z^2)}{x+y} \cdot c^2 \geq 4\sqrt{3}F; \forall x, y, z \in (0, \infty)$$

Proposed by D.M.Bătinețu-Giurgiu,Dan Nănuți-Romania

S.2106 Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that:

$$1+f(x+y)\le f(x)+f(y)\le x+y+2;\forall x,y\in\mathbb{R}$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

S.2107 Let $f:\mathbb{R}_+^*\to(0,\infty), m\ge 0, x,y,z>0, f(t)=\frac{t^4+1}{\sqrt{t^4-t^2+1}}$ then in ΔABC holds:

$$\left(\frac{f(x+y)}{z}\cdot ab\right)^{m+1}+\left(\frac{f(y+z)}{x}\cdot bc\right)^{m+1}+\left(\frac{f(z+x)}{y}\cdot ca\right)^{m+1}\ge 16^{m+1}\left(\sqrt{3}\right)^{1-m}F^{m+1}$$

Proposed by D.M.Bătinețu-Giurgiu, Flaviu Cristian Verde-Romania

S.2108 Let $A_1A_2\dots A_n, n\ge 3$ convex polygon with sides $a_k=A_kA_{k+1}, k=\overline{1,n}, A_{n+1}=A_1$, then:

$$\sum_{k=1}^{n}\frac{a_k^{2m+4}+a_{k+1}^{2m+4}}{a_ka_{k+1}}\ge\frac{2^{2m+3}}{n^m}\cdot F^{m+1}\cdot\tan^{m+1}\frac{\pi}{n};\forall m\ge 0$$

Proposed by D.M.Bătinețu-Giurgiu, Flaviu Cristian Verde-Romania

S.2109 If $x,y,a_k\in\mathbb{R}_+^*,\forall k=\overline{1,n}, n\in\mathbb{N}, n\ge 2$ and $a_1a_2\dots a_n=1, a_{n+1}=a_1, a_{n+2}=a_2$, then:

$$\sum_{k=1}^{n}\left(x\cdot a_k+y\cdot\frac{a_{k+1}}{a_{k+2}}\right)^3\ge 8n\sqrt{x^3y^3}$$

Proposed by D.M.Bătinețu-Giurgiu,Claudia Nănuți-Romania

S.2110 If $n\in\mathbb{N}, n\ge 2$ and $a_k\in\mathbb{R}^*,\forall k=\overline{1,n}$, then:

$$\frac{a_1^2}{a_2^2}+2\cdot\sqrt{\frac{a_2^2}{a_3^2}}+3\cdot\sqrt{\frac{a_3^2}{a_4^2}}+\dots+(n-1)\cdot\sqrt[n-1]{\frac{a_{n-1}^2}{a_n^2}}+n\cdot\sqrt[n]{\frac{a_n^2}{a_1^2}}\ge\frac{n(n+1)}{2}$$

Proposed by D.M.Bătinețu-Giurgiu,Claudia Nănuți-Romania

S.2111 Let $m,x,y\ge 0, x+y.0$ then in ΔABC the following relationship holds:

$$\frac{a^{m+2}}{(ax+by)^m}+\frac{b^{m+2}}{(bx+cy)^m}+\frac{c^{m+2}}{(cx+ay)^m}\ge\frac{4\sqrt{3}}{(x+y)^m}\cdot F$$

Proposed by D.M.Bătinețu-Giurgiu,Dan Nănuți-Romania

S.2112 If $x,y>0$, then in ΔABC the following relationship holds:

$$\frac{x^2a^4+y^2b^4}{bc}+\frac{x^2b^4+y^2c^4}{ca}+\frac{x^2c^4+y^2a^4}{ca}\ge 2\sqrt{3}(x+y)^2\cdot F$$

Proposed by D.M.Bătinețu-Giurgiu,Dan Nănuți-Romania

S.2113 If $x, y, z > 0$ then in ΔABC the following relationship holds:

$$\frac{x^3a^5}{(y+z)h_a^3}+\frac{y^3b^5}{(z+x)h_b^3}+\frac{z^3c^5}{(x+y)h_c^3}\geq\frac{16}{9}(xy+yz+zx)\cdot F$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

S.2114 In ΔABC the following relationship holds:

$$a^4+b^4+c^4\geq 16F^2+\frac{1}{2F^2}\cdot\sum_{cyc}\left(\frac{a}{h_a}-\frac{b}{h_b}\right)^2$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

S.2115 In ΔABC the following relationship holds: $a(s-a)+b(s-b)+c(s-c)\geq 2\sqrt{3}\cdot F$

Proposed by D.M.Bătinețu-Giurgiu, Florică Anastase-Romania

S.2116 In ΔABC the following relationship holds:

$$\frac{a^4}{h_bh_c}+\frac{b^4}{h_ch_a}+\frac{c^4}{h_ah_b}\geq\frac{16}{\sqrt{3}}\cdot F$$

Proposed by D.M.Bătinețu-Giurgiu, Florică Anastase-Romania

S.2117 Let $f:\mathbb{R}\to\mathbb{R}$ such that $1+f(x+y),=f(x)+f(y)\leq x+y+2, \forall x,y\in\mathbb{R}$, then:

$$\frac{f(x^2)}{y+f(z^2)}+\frac{f(y^2)}{z+f(x^2)}+\frac{f(z^2)}{x+f(y^2)}\geq 2; \forall x,y,z>0$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

S.2118 Let $m\geq 0, A_1A_2\dots A_n, n\geq 3$ convex polygon with area F and sides $a_k=A_kA_{k+1}, k=\overline{1,n}, A_{n+1}=A_1, A_{n+2}=A_2$, then:

$$\sum_{k=1}^{n}\frac{a_k^{2m+4}+a_{k+1}^{2m+4}+a_{k+2}^{2m+4}}{\sqrt[3]{a_k^2a_{k+1}^2a_{k+2}^2}}\geq\frac{3\cdot 4^{m+1}}{n^m}\cdot F^{m+1}\cdot\tan^{m+1}\frac{\pi}{n}$$

Proposed by D.M.Bătinețu-Giurgiu,Claudia Nănuți-Romania

S.2119 Let $M\in Int(\Delta ABC), d_a=d(M,BC), d_b=d(M,CA), d_c=d(M,AB)$, then:

$$\frac{x^2a^3}{(y+z)^2d_a}+\frac{y^2b^3}{(z+x)^2d_c}+\frac{z^2c^3}{(x+y)^2d_c}\geq 6F; \forall x,y,z>0$$

Proposed by D.M.Bătinețu-Giurgiu,Claudia Nănuți-Romania

S.2120 If $a,b>0$ then: $(a+b)^{2\sqrt{ab}}\leq 4^{\sqrt{ab}}\cdot\left(\sqrt{ab}\right)^{a+b}$

Proposed by Daniel Sitaru, Claudia Nănuți -Romania

S.2121 Let $f:(0,\infty)\to(0,\infty), f(x)=\frac{x^4+1}{\sqrt{x^4-x^2+1}}$, then in ΔABC holds:

$$\frac{af(x)}{h_a}+\frac{bf(y)}{h_b}+\frac{cf(z)}{h_c}\geq 4\sqrt{xy+yz+zx};\forall x,y,z>0$$

Proposed by D.M.Bătinețu-Giurgiu,Florică Anastase-Romania

S.2122 If $a,b,c>0, abc=1$ then:

$$\left(\sum_{cyc}a\right)^2\cdot\sum_{cyc}\frac{1}{a+b^2+c^7}\leq 3+\sum_{cyc}\left(a+\frac{1}{a}\right)$$

Proposed by Daniel Sitaru, Claudia Nănuți -Romania

S.2123 In acute ΔABC holds:

$$\prod_{cyc}(1+\tan A\cot B)\geq 2+32F^2\prod_{cyc}\sqrt[3]{\frac{1}{(b^2+c^2-a^2)^2}}$$

Proposed by Daniel Sitaru, Claudia Nănuți-Romania

S.2124 Let $f:(0,\infty)\to(0,\infty), f(t)=\frac{t^4+1}{\sqrt{t^4-t^2+1}}$, then in ΔABC holds:

$$a^2f(x)+b^2f(y)+c^2f(z)\geq 4F\cdot\sqrt{\frac{xy}{\sin^2\frac{C}{2}}+\frac{yz}{\sin^2\frac{A}{2}}+\frac{zx}{\sin^2\frac{B}{2}}};\forall x,y,z>0$$

Proposed by D.M.Bătinețu-Giurgiu,Dan Nănuți-Romania

S.2125 Let $f:(0,\infty)\to(0,\infty), f(x)=\frac{x^4+1}{\sqrt{x^4-x^2+1}}$, then in ΔABC holds:

$$\frac{a^2f(x)}{y+z}+\frac{b^2f(y)}{z+x}+\frac{c^2f(z)}{x+y}\geq 4\sqrt{3}\cdot F;\forall x,y,z>0$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

S.2126 In ΔABC the following relationship holds:

$$\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\geq 2\cdot\sqrt[4]{27}\cdot\sqrt{F}$$

Proposed by D.M.Bătinețu-Giurgiu, Gabriela Bondoc-Romania

S.2127 If $x,y,z\geq 0$ then in ΔABC the following relationship holds:

$$\frac{e^x+y+1}{z+1}\cdot ab+\frac{e^y+z+1}{x+1}\cdot bc+\frac{e^z+x+1}{y+1}\cdot ca\geq 8\sqrt{3}\cdot F$$

Proposed by D.M.Bătinețu-Giurgiu, Gabriela Bondoc -Romania

S.2128 If $x, y, z > 0$ then in ΔABC the following relationship holds:

$$\frac{a^2 e^{x^2}}{y+z} + \frac{b^2 e^{y^2}}{z+x} + \frac{c^2 e^{z^2}}{x+y} > 4\sqrt{3} \cdot F$$

Proposed by D.M.Bătinețu-Giurgiu, Gabriela Bondoc -Romania

S.2129 Let $A_1 A_2 \dots A_n$ convex polygon with area F and the sides $a_k = A_k A_{k+1}$,

$$\forall k = \overline{1,n}, A_{n+1} = A_1, A_{n+2} = A_2, A_{n+3} = A_3.$$

If $B_k = \frac{a_k^3}{a_k F_m^2 + a_{k+1} F_{m+1}^2} + \frac{a_{k+1}^3}{a_{k+1} F_m^2 + a_{k+2} F_{m+1}^2} + \frac{a_{k+2}^2}{a_{k+2} F_m^2 + a_k F_{m+1}^2}, \forall k = \overline{1,n}$. Then:

$$\sum_{k=1}^{n} B_k \geq \frac{12F}{F_{2m+1}} \cdot \tan\frac{\pi}{m}; \forall m \in \mathbb{N}^*$$

Proposed by D.M.Bătinețu-Giurgiu,Claudia Nănuți-Romania

S.2130 If $n \in \mathbb{N}, n \geq 2$ and $a_k \in (0, \infty), \forall k = \overline{1,n}$, then:

$$\frac{a_1}{a_2} + 4 \cdot \sqrt[4]{\frac{a_2}{a_3}} + 9 \cdot \sqrt[9]{\frac{a_3}{a_4}} + \dots + n^2 \cdot \sqrt[n^2]{\frac{a_n}{a_1}} \geq \frac{n(n+1)(2n+1)}{6}$$

Proposed by D.M.Bătinețu-Giurgiu,Claudia Nănuți-Romania

S.2131 If $m \geq 0$ and $x, y, z \geq 0$, then:

$$\frac{x^{m+2}}{\left(y + \sqrt{yz} + \sqrt{zx}\right)^m} + \frac{y^{m+2}}{\left(z + \sqrt{zx} + \sqrt{xy}\right)^m} + \frac{z^{m+2}}{\left(x + \sqrt{xy} + \sqrt{yz}\right)^m} \geq \frac{1}{3^m}(x^2 + y^2 + z^2)$$

Proposed by D.M.Bătinețu-Giurgiu,Claudia Nănuți-Romania

S.2132 In ΔABC the following relationship holds:

$$\frac{a^3}{\frac{1}{h_b} + \frac{1}{\sqrt[3]{h_a h_c^2}}} + \frac{b^3}{\frac{1}{h_c} + \frac{1}{\sqrt[3]{h_b h_c^2}}} + \frac{c^3}{\frac{1}{h_a} + \frac{1}{\sqrt[3]{h_c h_b^2}}} \geq 4\sqrt{3} \cdot F^2$$

Proposed by D.M.Bătinețu-Giurgiu,Claudia Nănuți-Romania

S.2133 If $x, y, z > 0$, then in ΔABC the following relationship holds:

$$\frac{x^2 a^7}{(y+z)^2 h_a} + \frac{y^2 b^7}{(z+x)^2 h_b} + \frac{z^2 c^7}{(x+y)^2 h_c} \geq \frac{32}{3} F^3$$

Proposed by D.M.Bătinețu-Giurgiu,Claudia Nănuți-Romania

S.2134 Let $x, y > 0, \Delta A_iB_iC_i, i = 1,2$ triangles with areas $F_i, i = 1,2$ and altitudes $h_{a_i}, h_{b_i}, h_{c_i}, i = 1,2$, then holds:

$$\frac{a_1^{1-x}a_2^{1-y}}{h_{a_1}^x h_{a_2}^y} + \frac{b_1^{1-x}b_2^{1-y}}{h_{b_1}^x h_{b_2}^y} + \frac{c_1^{1-x}c_2^{1-y}}{h_{c_1}^x h_{c_2}^y} \geq 2^{2-x-y}\sqrt{3} \cdot F_1^{\frac{1}{2}-x} F_2^{\frac{1}{2}-y}$$

Proposed by D.M.Bătinețu-Giurgiu, Gabriela Bondoc-Romania

S.2135 Let $x, y, z > 0$, n_a –Nagel's cevian and g_a –Gergonne's cevian, the following relationship holds:

$$\frac{x \cdot \sqrt[3]{n_a g_a}}{y+z}\left(\frac{a}{\sqrt[3]{h_a}}\right)^2 + \frac{y \cdot \sqrt[3]{n_b g_b}}{z+x}\left(\frac{b}{\sqrt[3]{h_b}}\right)^2 + \frac{z \cdot \sqrt[3]{n_c z_c}}{x+y}\left(\frac{c}{\sqrt[3]{h_c}}\right)^2 \geq 2\sqrt{3} \cdot F$$

Proposed by D.M.Bătinețu-Giurgiu,Dan Nănuți-Romania

S.2136 In ΔABC the following relationship holds:

$$\frac{w_a}{m_a} + \frac{w_b}{m_b} + \frac{w_c}{m_c} \leq \frac{1}{s}\sqrt{\left(\frac{4R}{r}+1\right)\left(\sum_{cyc} h_a^2 + \frac{s^2(R^2-4r)}{2r^2}\right)}$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

S.2137 In ΔABC the following relationship holds:

$$\frac{w_a}{h_a} + \frac{w_b}{h_b} + \frac{w_c}{h_c} \geq 3\sqrt{\frac{2r}{R}}$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

S.2138 In ΔABC the following relationship holds:

$$3s^2 \leq 3s^2 + \frac{4r(R-2r)}{5} \leq (4R+r)^2$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

S.2139 In ΔABC the following relationship holds:

$$\sum_{cyc} m_a^2 + r(R-2r) \leq \sum_{cyc} r_a^2$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

S.2140 In ΔABC the following relationship holds:

$$\sum_{cyc} r_a^2 \leq \sum_{cyc} m_a^2 + 16(R^2-4r^2)$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

S.2141 In $\Delta ABC, p_a$ −Spieker's cevian, prove that:

$$\sum_{cyc} n_a^2 \geq \sum_{cyc} p_a^2 + 27r^2 \sum_{cyc} \left(\frac{b-c}{s+a}\right)^2$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

S.2142 Find all continuous functions $f: \mathbb{R} \to \mathbb{R}$ such that $f(\alpha x) = f(\beta x) - x^{\alpha\beta}$,

$$\forall x \in \mathbb{R}, \alpha, \beta > 0.$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

S.2143 Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that

$$f(x+2y) = f(x^3)f(x-2y) + \alpha\beta xy, \forall x, y \in \mathbb{R}, \alpha, \beta \geq 0.$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

S.2144 Let $a, b \in \mathbb{R}$ such that

$$\left| x \int_{-1}^{x} (at+b)dt \right| \sqrt{1-x^2} \leq 1; \forall |x| \leq 1$$

Prove that: $|a| \leq 16$.

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

S.2145 Find all positive real numbers α such that:

$$\sum_{cyc} r_a^2 \leq \sum_{cyc} m_a^2 + \alpha r(R-2r) \leq \sum_{cyc} r_a^2, \forall \Delta ABC$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

S.2146 In ΔABC the following relationship holds:

$$\frac{h_a^2}{a(b+c-a)} + \frac{h_b^2}{b(c+a-b)} + \frac{h_c^2}{c(a+b-c)} \leq \frac{ab+bc+ca}{8Rr} - \frac{1}{2}\sum_{cyc} \frac{(n_a-g_a)^2}{a^2}$$

Proposed by Soumava Chakraborty-India

S.2147 Let $0 < \alpha < 2$. Solve for real numbers:

$$\begin{cases} \alpha x + 2y + 3z = 2020 \\ 3x + \alpha y + 2z = 2021 \\ 2x + 3y + \alpha z = 2022 \end{cases}$$

Proposed by Nguyen Van Canh- Vietnam

S.2148 In ΔABC the following relationship holds:

$$ab+bc+ca\geq\sum_{cyc}\frac{a^3(b+c-a)}{(b+c)^2}$$

Proposed by Nguyen Van Canh- Vietnam

S.2149 In ΔABC the following relationship holds:

$$\sum_{cyc}h_a+\frac{2r(R-2r)}{R}\leq\sum_{cyc}r_a$$

Proposed by Nguyen Van Canh- Vietnam

S.2150 In ΔABC the following relationship holds: $\sum_{cyc}h_a+\frac{r(R-2r)}{5R}\leq\sum_{cyc}m_a$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

S.2151 Let $\alpha\geq 6$. In ΔABC the following relationship holds:

$$\sum_{cyc}w_a^2+\alpha(R^2-4r^2)\geq\sum_{cyc}m_a^2$$

Proposed by Nguyen Van Canh- Vietnam

S.2152 Let $\alpha\geq 6$. In ΔABC the following relationship holds:

$$\sum_{cyc}h_a^2+\alpha(R^2-4r^2)\geq\sum_{cyc}w_a^2$$

Proposed by Nguyen Van Canh- Vietnam

S.2153 In ΔABC the following relationship holds:

$$\sum_{cyc}\max\{w_a^2,s(s-a)\}\leq\frac{(4R+r)^2}{3}$$

Proposed by Nguyen Van Canh-Vietnam

S.2154 In ΔABC the following relationship holds:

$$\sum_{cyc}h_a^2+3r(R-2r)\leq\min\left\{\sum_{cyc}r_a^2,\sum_{cyc}m_a^2\right\}$$

Proposed by Nguyen Van Canh-Vietnam

S.2155 In ΔABC the following relationship holds:

$$\sum_{cyc}w_a^2+2r(R-2r)\leq\min\left\{\sum_{cyc}r_a^2,\sum_{cyc}m_a^2\right\}$$

Proposed by Nguyen Van Canh-Vietnam

S.2156 In ΔABC the following relationship holds:

$$9\min\{a^2,b^2,c^2\}\le 4\max\left\{\sum_{cyc} m_a^2,\sum_{cyc} r_a^2\right\}$$

Proposed by Nguyen Van Canh-Vietnam

S.2157 Let $a,b,c>0$ such that $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\le\sum a^3$. Prove that:

$$\frac{9\sum a^2}{(\sum a)^2}\ge(\sum ab)(\sum a)-6abc$$

Proposed by Nguyen Van Canh-Vietnam

S.2158 In ΔABC the following relationship holds:

$$\sum_{cyc} m_a r_a\le w_a^2+w_b^2+w_c^2+\frac{15s^2(R^4-16r^4)}{r^4}$$

Proposed by Nguyen Van Canh-Vietnam

S.2159 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{r_a^2}{r_b+r_c}\ge r_a+r_b+r_c+\frac{r^2(R-2r)}{10(R^2+16r^2)}$$

Proposed by Nguyen Van Canh-Vietnam

S.2160 Find all functions $f:\mathbb{Z}\to\mathbb{R}$ such that $f(x^{2021}+1)=f(y^{2021})-2xy,\forall x,y\in\mathbb{Z}$.

Proposed by Nguyen Van Canh-Vietnam

S.2161 In $\Delta ABC, p_a$ –Spieker's cevian, n_a –Nagel's cevian, the following relationship holds:

$$\sum_{cyc}(n_a+p_a)\le\sqrt{3}\left(\sqrt{\sum_{cyc} m_a^2+\frac{s^2(R^2-4r^2)}{5r^2}}+\sqrt{\sum_{cyc} s_a^2+\frac{\sqrt{3}s^2(R^4-16r^2)}{2r^2}}\right)$$

Proposed by Nguyen Van Canh-Vietnam

S.2162 In ΔABC the following relationship holds: $\sum_{cyc} r_a r_b+r(R-2r)\le\sum_{cyc} r_a^2$

Proposed by Nguyen Van Canh-Vietnam

S.2163 In ΔABC the following relationship holds:

$$\sum_{cyc} r_a^2\le\sum_{cyc} r_a r_b+16(R^2-4r^2)$$

Proposed by Nguyen Van Canh-Vietnam

S.2164 In ΔABC the following relationship holds:

$$\sum_{cyc} m_a^2 m_b^2 + \frac{r^3(R-2r)}{4} \le \sum_{cyc} m_a^4$$

Proposed by Nguyen Van Canh-Vietnam

S.2165 In ΔABC the following relationship holds:

$$\sum_{cyc}(n_a+g_a)^4 - 9\sum_{cyc} a^4 + \sum_{cyc}\frac{(b-c)^4(b+c)^2}{a^2} \ge 16\sum_{cyc} m_a^2\big(m_a^2 - s(s-a)\big)$$

Proposed by Soumava Chakraborty-India

S.2166 In ΔABC the following relationship holds:

$$\frac{s}{8Rr}\sum_{cyc} a h_a w_a(b^2+c^2) + \sum_{cyc} w_a^2 r_a^2 \le s^2(s^2+4Rr+r^2)$$

Proposed by Soumava Chakraborty-India

S.2167 Let $\alpha,\beta,\gamma \in \mathbb{R}$ such that $|\alpha x^3 + \beta x^2 + \gamma x| \le 1, \forall |x| \le 1$. Prove that:

$$a)\ \sum_{cyc}|\alpha|^3|\beta-\gamma| \ge |\alpha+\beta+\gamma|\prod_{cyc}|\alpha-\beta|$$

$$b)\ |3\alpha+2\beta+\gamma| \le 9$$

Proposed by Nguyen Van Canh-Vietnam

S.2168 Let $a,b,c \ge 0, a+b+c=3$. Prove that:

$$\frac{a+1}{\sqrt{a^2+3}+4bc} + \frac{b+1}{\sqrt{b^2+3}+4ca} + \frac{c+1}{\sqrt{c^2+3}+4ab} \ge 1$$

Proposed by Phan Ngoc Chau-Vietnam

S.2169 Let $a,b,c \ge 0, ab+bc+ca > 0$ and $a+b+c=3$. Find Max value of the following expression:

$$T = \sqrt{\frac{a^2+b^2}{a+b}} + \sqrt{\frac{b^2+c^2}{b+c}} + \sqrt{\frac{c^2+a^2}{c+a}}$$

Proposed by Phan Ngoc Chau-Vietnam

S.2170 Let $a,b,c > 0$. Prove that:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+k\frac{ab+bc+ca}{a^2+b^2+c^2} \ge k+3, \forall k \in \left(0,\frac{3}{2}\right]$$

Proposed by Phan Ngoc Chau-Vietnam

S.2171 In $\Delta ABC, AE$ —internal bisector, d —antiparallel through the incenter, $d \cap [AB] = \{K\}, d \cap [AC] = \{L\}$. Prove that:

$$\frac{KI}{AE} + \frac{LI}{CE} \geq \frac{b+c}{s}$$

Proposed by Mehmet Șahin-Turkiye

S.2172 If $r_0 = 1, r_1 = 11, r_2 = 111, r_3 = 1111, \dots, n_1 < n_2 < n_3 < \cdots, k = 2^{n_1} + 2^{n_2} + 2^{n_3} + \cdots + 2^n$ and $a_1 < a_2 < a_3 < \cdots, a_k = r_{n_1} + r_{n_2} + r_{n_3} + \cdots + r_n$, then find a_{2019}.

Proposed by Neculai Stanciu-Romania

S.2173 In acute $\Delta ABC, AE$ —internal bisector, d —antiparallel through the incenter $d \cap [AB] = \{K\}, d \cap [AC] = \{L\}, BN \perp d, CM \perp d$. Prove that: $BN + CM \leq 2(n_a - R)$

Proposed by Mehmet Șahin-Turkiye

S.2174 Solve for real numbers: $(2x+1)\left(1+\sqrt{(2x+1)^2+7}\right) + x\left(1+\sqrt{x^2+7}\right) = 0$

Proposed by Babek Ceferov-Azerbaijan

S.2175 In ΔABC the following relationship holds:

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + \frac{R^3 - 8r^3}{2r^3} \geq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} + \frac{3}{2}$$

Proposed by Nguyen Van Canh-Vietnam

S.2176 Let $a, b, c > 0$ such that $a^{2021} + b^{2021} + c^{2021} = 4$. Find the maximum value of

$$P = a + b + c + (abc)^{2021} + ab + bc + ca$$

Proposed by Nguyen Van Canh-Vietnam

S.2177 In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{w_a}{2w_a^3 + w_b^3} \leq \frac{1}{9r^2}$$

Proposed by Marin Chirciu-Romania

S.2178 Determine all pairs (x, y) of integers which satisfy $x^6 + x^3 - y^4 + 1 = 0$.

Proposed by Neculai Stanciu-Romania

S.2179 In ΔABC the following relationship holds:

$$\left(\frac{r_a}{r_b} + \frac{r_b}{r_c} + \frac{r_c}{r_a}\right)\left(\frac{r_b}{r_a} + \frac{r_c}{r_b} + \frac{r_a}{r_c}\right) \geq 1 + \frac{4R}{r}$$

Proposed by Daniel Sitaru-Romania

S.2180 If $a, b > 0$ then:

$$\frac{2a}{2a+3b}\log\left(1+\frac{3b}{2a}\right)+\frac{3b}{2a+3b}\log\left(1+\frac{2a}{3b}\right)\leq \log 2$$

Proposed by Daniel Sitaru-Romania

S.2181 If $a, b, c > 0, a+b+c=3$ then:

$$\frac{ac^3}{(c^4+1)(c^2+c+1)}+\frac{ba^3}{(a^4+1)(a^2+a+1)}+\frac{cb^3}{(b^4+1)(b^2+b+1)}\leq\frac{1}{2}$$

Proposed by Daniel Sitaru-Romania

S.2182 If $a, b, c > 0, a+b+c=1$ then: $\sum_{cyc}(a+b)\sqrt[a+b]{a^a b^b}\leq a^2+b^2+c^2+a^a b^b c^c$

Proposed by Daniel Sitaru-Romania

S.2183 In $\Delta ABC, \Gamma$ –Toricelli's point, holds:

$$(\Gamma\mathrm{A}^3+\Gamma\mathrm{B}^3+\Gamma\mathrm{C}^3)\left(\frac{1}{\Gamma\mathrm{A}}+\frac{1}{\Gamma\mathrm{B}}+\frac{1}{\Gamma\mathrm{C}}\right)\geq 36r^2$$

Proposed by Daniel Sitaru-Romania

S.2184 If $a, b, c, d, e, f > 0$ then:

$$3abcdef(abc+def-5)+(ab+bc+ca)(de+ef+fd)\geq 0$$

Proposed by Daniel Sitaru-Romania

S.2185 Find:

$$\Omega=\int_0^{\frac{\pi}{2}}\frac{\cos x\cdot\sinh x}{5+e^x\sin\left(x+\frac{\pi}{4}\right)+e^{-x}\cos\left(x+\frac{\pi}{4}\right)}dx$$

Proposed by Daniel Sitaru-Romania

S.2186 Find:

$$\Omega=\int\frac{\sin x+\sqrt{3}\cos x}{\sin(3x)}dx$$

Proposed by Daniel Sitaru, Claudia Nănuți-Romania

S.2187 If $a, x > 0, b, c, y, z\in\mathbb{R}$ then:

$$\frac{(a+x)^2-(b+y)^2-(c+z)^2}{a+x}\geq\frac{a^2-b^2-c^2}{a}+\frac{x^2-y^2-z^2}{x}$$

Proposed by Daniel Sitaru, Claudia Nănuți-Romania

S.2188 If $a, b, c, d \in \mathbb{C}$ then:

$$\frac{1}{\sqrt{7}} \cdot |a + b + c + d| \leq \sqrt{|a|^2 + \frac{1}{6}|b|^2} + \sqrt{|c|^2 + \frac{1}{6}|d|^2}$$

Proposed by Daniel Sitaru, Dan Nănuți-Romania

S.2189 If $a, b, c, d > 0, 0 \leq x \leq 1$ then:

$$\left((1-x)a + xc\right)^2 - \left((1-x)b + xd\right)^2 \geq (a^2 - b^2)^{1-x}(c^2 - d^2)^x$$

Proposed by Daniel Sitaru, Dan Nănuți-Romania

S.2190 Solve for real numbers: $\begin{cases} x + y = 4 \\ 2|x - y| = (|x| + |y|) \cdot \left|\frac{x}{|x|} - \frac{y}{|y|}\right| \end{cases}$

Proposed by Daniel Sitaru-Romania

S.2191 $A = \begin{pmatrix} 2 & 1+i \\ 1-i & 3 \end{pmatrix}, B = \begin{pmatrix} 3 & 1-i \\ 1+i & 2 \end{pmatrix}, i^2 = -1$. Prove that:

$$(x, y)\left(\frac{1}{2}(A + B) - 2(A^{-1} + B^{-1})^{-1}\right)\begin{pmatrix} x \\ y \end{pmatrix} \geq 0, x, y \in \mathbb{R}$$

Proposed by Daniel Sitaru-Romania

S.2192 If $a \geq 0$ then:

$$\frac{a}{2}\int_0^a x(\tan^{-1} x)^{12}\, dx < \int_0^a x^2(\tan^{-1} x)^{12}\, dx < \frac{3a}{4}\int_0^a x(\tan^{-1} x)^{12}\, dx$$

Proposed by Daniel Sitaru-Romania

S.2193 $M = \left\{\begin{pmatrix} a & b \\ \bar{b} & a \end{pmatrix} \middle| a > 0, b \in \mathbb{C}\right\}$. If $X, Y, Z \in M$ then:

$$Tr(XY) + Tr(YZ) + Tr(ZX) \leq Tr(X^2) + Tr(Y^2) + Tr(Z^2)$$

Proposed by Daniel Sitaru,Gabriela Bondoc-Romania

S.2194 $M = \left\{\begin{pmatrix} a & b+ic \\ b-ic & a \end{pmatrix} \middle| a > 0, b, c \in \mathbb{R}, i^2 = -1\right\}$.If $X, Y, z \in M$ then:

$$Tr(XY) + Tr(YZ) + Tr(ZX) \leq \sqrt{Tr(X^2)Tr(Y^2)} + \sqrt{Tr(Y^2)Tr(Z^2)} + \sqrt{Tr(Z^2)(Tr(X^2)}$$

Proposed by Daniel Sitaru,Gabriela Bondoc-Romania

S.2195 Let $a, b, c \geq 0, a + b + c = 3$. Prove that:

$$\frac{ab(b+1)}{c+1} + \frac{bc(c+1)}{a+1} + \frac{ca(a+1)}{b+1} \geq ab + bc + ca$$

Proposed by Phan Ngoc Chau-Vietnam

S.2196 Let $a,b,c>0, abc=1$. Prove that:

$$2\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)+7(a+b+c)\geq 9\left(\sqrt[3]{\frac{5a+4b}{9b}}+\sqrt[3]{\frac{5b+4c}{9c}}+\sqrt[3]{\frac{5c+4a}{9a}}\right)$$

Proposed by Phan Ngoc Chau-Vietnam

S.2197 In ΔABC the following relationship holds:

$$A^2+B^2+C^2\geq\frac{3\pi^2}{8}$$

Proposed by Phan Ngoc Chau-Vietnam

S.2198 Let $a,b,c>0, ab+bc+ca=2(a+b+c)$. Prove that:

$$\sqrt{a+b+\frac{c}{ab}}+\sqrt{b+c+\frac{a}{bc}}+\sqrt{c+a+\frac{b}{ca}}+\sqrt{abc}\left(\frac{3}{a+b+c}-2\right)\geq\frac{a+b+c}{\sqrt{abc}}$$

Proposed by Phan Ngoc Chau-Vietnam

S.2199 In $\Delta ABC, R_a, R_b$ and R_c are the circumradius of $\Delta BIC, \Delta AIC$ and ΔAIB respectively. Prove that:

$$a\cdot\frac{r_a}{R_a}+b\cdot\frac{r_b}{R_b}+c\cdot\frac{r_c}{R_c}\leq\frac{9\sqrt{3}}{2}R$$

Proposed by Ertan Yildirim- Turkiye

S.2200 In ΔABC the following relationship holds:

$$\frac{R}{r}\geq\sqrt{\frac{(r_a+r_b+r_c)^3}{(m_a+m_b+m_c)^2(h_a+h_b+h_c)}}\cdot\max\left\{\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}};\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}};\sqrt{\frac{a}{c}}+\sqrt{\frac{c}{a}}\right\}$$

Proposed by Bogdan Fuștei-Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

UNDERGRADUATE PROBLEMS

U.1602 Let the differential equation $(1+x)y''(x)+(1-x)y'(x)=\frac{1-x}{1+x}y(x), y(0)=1, y'(0)=0$ then prove that:

$$\int_0^{\infty}\left(y''(x)+y'(x)+y(x)\right)e^{-x}dx=\frac{3}{2}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.1603 Find a closed form:

$$\Omega(a)=\int_0^{\infty}\frac{x}{(x^2+x+1)(1+a^2x^2)}dx\,, a\in\mathbb{R}-\{0\}$$

Proposed by Vasile Mircea Popa-Romania

U.1604 Find all $x,y,z\geq 0$ such that: $x+y+z=\sqrt[4]{4xyz}\left(\sqrt[4]{x}+\sqrt[4]{y}\right)$

Proposed by Daniel Sitaru-Romania

U.1605 Let the differential equation $y''(x)+y(x)=xy^{(3)}(x), y(0)=1, y'(0)=0, y''(0)=-1, y^{(3)}(0)=0$, then show that:

$$\int_{-\infty}^{\infty}y(x)e^{-\frac{x^2}{2}}dx=\frac{\sqrt{2\pi}}{e}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.1606 If $0\leq x,y,z\leq\frac{\pi}{4}$ then:

$$2\cos^2 x\cdot\cos^2 y\cdot\cos^2 z\leq 1+\cos 2x\cos 2y\cos 2z\leq 8\cos^2 x\cos^2 y\cos^2 z$$

Proposed by Daniel Sitaru-Romania

U.1607 If $a,b,c,d,x,y,z,t>0$ then:

$$\left(\frac{a}{x}\right)^{3a}\left(\frac{b}{y}\right)^{3b}\left(\frac{c}{z}\right)^{2c}\left(\frac{d}{t}\right)^{d}\geq\left(\frac{a+b}{x+y}\right)^{a+b}\left(\frac{a+b+c}{x+y+z}\right)^{a+b+c}\left(\frac{a+b+c+d}{x+y+z+t}\right)^{a+b+c+d}$$

Proposed by Daniel Sitaru-Romania

U.1608 Find:

$$\Omega=\int_1^{21}\frac{dx}{e^{\left[2x+\frac{1}{4}\right]}};[*]-\text{GIF}.$$

Proposed by Daniel Sitaru,Claudia Nănuți-Romania

U.1609 Find:

$$\Omega=\lim_{n\to\infty}\frac{H_{n+1}H_n^4+H_{n+2}H_{n+1}^4+H_nH_{n+2}^4}{n(H_nH_{n+1}^4+H_{n+1}H_{n+2}^4+H_{n+2}H_n^4)}$$

Proposed by Daniel Sitaru, Claudia Nănuți-Romania

U.1610 If $0<a\le b$ then:

$$2\int_a^b\int_a^b\int_a^b\left(\frac{y+x}{y+z}+\frac{y+z}{y+x}\right)dxdydz+2(b-a)^3\le 3(b+a)(b-a)^2\log\left(\frac{b}{a}\right)$$

Proposed by Daniel Sitaru-Romania

U.1611 Let the differential equation $y''(x)+y(x)=y^{(3)}(x), y(0)=1, y'(0)=0, y''(0)=-1$, then prove that for $n\ge 2$:

$$\int_0^\infty y(x)e^{-nx}dx=\frac{n^2+n-1}{n^3+2n^2+n-1}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.1612 Find:

$$\Omega=\lim_{n\to\infty}\frac{H_n}{n(H_{2n-1}-2H_{n-1})}$$

Proposed by Daniel Sitaru-Romania

U.1613 If $0<a\le 1\le b$ then: $(a+b-1)^{a+b-1}+1\le a^a+b^b$

Proposed by Daniel Sitaru-Romania

U.1614 Find:

$$\Omega=\int_0^\infty\frac{x\log(1+x)}{(x+1)(x^2+1)}dx$$

Proposed by Vasile Mircea Popa-Romania

U.1615 If we define the function for $n\ge 1$

$$\phi(n)=\sum_{m=1}^{n}\sin^{(-1)^{\frac{1}{2}m(m+1)}}\left(\frac{\pi m}{4}\right)\sin^{\frac{1}{2}m(m-1)}\left(\frac{\pi m}{2}\right)$$

then prove that:

$$\sum_{n=1}^{\infty} 2^{-\frac{n}{2}}\phi(n)=\frac{3}{5}\left(2+\sqrt{2}\right)$$

Proposed by Srinivasa Raghava-AIRMC-India

U.1616 Prove that:

$$\int_0^{\infty}\left(\sum_{n=0}^{\infty} x^{n-1}\sin^{(-1)^{\frac{1}{2}n(n+1)}}\left(\frac{\pi n}{x}\right)\right)dx=\pi$$

Proposed by Srinivasa Raghava-AIRMC-India

U.1617 For $k>0$, prove that:

$$\int_0^1 x^3\tan^{-1}\left(1+\frac{k}{x}\right)dx=$$

$$=k^4\left(\frac{\pi}{64}-\frac{1}{16}\tan^{-1}\left(\frac{k+2}{k}\right)\right)+\frac{k^3}{16}-\frac{k^2}{16}+\frac{k}{24}+\frac{1}{4}\tan^{-1}(k+1)$$

Proposed by Srinivasa Raghava-AIRMC-India

U.1618 If $ABCD$ is a bicentric quadrilateral with the inradius r, the circumradius R and the diagonals d_1, d_2, then prove the following inequality

$$\left(\frac{R}{r}\right)^{2k}\geq 2^{k-1}\left(\sqrt{\left(\frac{d_1}{d_2}\right)^{3k}}+\sqrt{\left(\frac{d_2}{d_1}\right)^{3k}}\right)$$

Proposed by Marius Drăgan, Neculai Stanciu-Romania

U.1619 In $\Delta ABC, p_a$ –Spieker's cevian, the following relationship relationship holds:

$$3(R-r)s^{22}-r\sum_{cyc} r_a^2\geq R(s_a^2-m_a^2+p_a w_a)$$

Proposed by Soumava Chakraborty- India

U.1619 In $\Delta ABC, p_a$ –Spieker's cevian, the following relationship holds:

$$\sum_{cyc} p_a^2(2s+a)^2\leq 12s^2(s^2-7Rr-3r^2)$$

Proposed by Soumava Chakraborty- India

U.1620 If $x,y,z\in\mathbb{R}_+^*$, then prove that in any triangle ABC holds:

$$\frac{1}{x+y\sin^2\frac{A}{2}+z\cos^2\frac{B}{2}}+\frac{1}{x+y\sin^2\frac{B}{2}+z^2\cos^2\frac{C}{2}}+\frac{1}{x+y\sin^2\frac{C}{2}+z\cos^2\frac{A}{2}}\geq$$

$$\geq\frac{18R}{3(3x+y+2z)R+(z-y)r}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

U.1621 Find the first triplets (s,t,n) of nonzero natural numbers $(n\in\mathbb{N}^*,s\in\mathbb{N}^*,t\in\mathbb{N}^*)$ such that $n=s^2=\frac{t(t+1)}{2}$, i.e. which are simultaneously perfect squares and triangular numbers.

Proposed by Neculai Stanciu-Romania

U.1622 If $t>0$, then compute:

$$\Omega=\lim_{n\to\infty}n^{1-t}\left((n+1)^t\left(\sqrt[n+1]{n+1}\right)^t-n^t\left(\sqrt[n]{n}\right)^t\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

U.1623 Determine all natural numbers which satisfy simultaneously the following conditions $a\leq c\leq b, a\leq d\leq b, a\leq\frac{c+d}{cd}\leq b$.

Proposed by Neculai Stanciu-Romania

U.1624 If M is a point in the interior triangle $ABC, A'=AM\cap BC, B'=BM\cap AC, C'=CM\cap AB$ then prove that:

$$\frac{AA'}{MA'}+\frac{BB'}{MB'}+\frac{CC'}{MC'}-\frac{AM\cdot BM\cdot CM}{A'M\cdot B'M\cdot C'M}=1$$

Proposed by Neculai Stanciu-Romania

U.1625 If $A=\begin{pmatrix}1&1&3\\1&5&1\\3&1&1\end{pmatrix}$ and $A^n=\begin{pmatrix}a_n&b_n&c_n\\b_n&d_n&b_n\\c_n&b_n&a_n\end{pmatrix}, \forall n\in\mathbb{N}^*$, then compute $\lim_{n\to\infty}\frac{a_n\cdot b_n}{c_n\cdot d_n}$.

Proposed by Neculai Stanciu-Romania

U.1626 Prove that:

$$3[(x^2+y^2+z^2)^2+xyz(x+y+z)]\geq 2[(xy+yz+zx)^2+3(x^2y^2+y^2z^2+z^2x^2)],$$

$\forall x,y,z$ real numbers.

Proposed by Neculai Stanciu-Romania

U.1627 If $a>0$ the prove the following inequalities:

$$(i)\ \frac{a+1}{2} \ge \frac{a(a^2+1)}{a^3+1};\ (ii)\ \frac{(a+1)}{2} \le \frac{a^3+1}{a^2+1};\ (iii)\frac{a^3+1}{a^2+1} \ge \sqrt{a^2-a+1} \ge \sqrt[4]{\frac{a^4+1}{2}}$$

Proposed by Neculai Stanciu-Romania

U.1628 Compute:

$$\lim_{n\to\infty} \frac{(n+4)[(n-1)!+2^{n-1}]-4n[(n-2)!+2^{n-2}]+(4n-8)[(n-3)!+2^{n-3}]}{n!}$$

Proposed by Neculai Stanciu-Romania

U.1629 If $(x_n)_{n\ge1}$is defined by $x_0 = a > 0$ and $x_{n+1} = \frac{-x_n+3}{-3x_n+5}, \forall n \in \mathbb{N}^*$, then find x_n.

Proposed by Neculai Stanciu-Romania

U.1630 If $n \in \mathbb{N}^*, x_k \in \mathbb{R}_+^*, k = \overline{1,n}, X_n(t) = \sum_{k=1}^n x_k^t, t \in \mathbb{R}, X_n(1) = \sum_{k=1}^n x_k = X_n, m \in [1,\infty)$ then show that $\sum_{k=1}^n x_k(X_n(-m) - x_k^{-m}) \ge \frac{(n-1)n^m}{X_n^{m-1}}$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

U.1631 Prove that in any right angled triangle is true the following inequality

$$8m_a m_b m_c \ge 5abc$$

Proposed by Neculai Stanciu-Romania

U.1632 Determine all the functions $f: \mathbb{N}^* \to [1,\infty)$ which verify the conditions

$$(i)\ f(2) = 2, (ii) f(n) \le f(n+1), (iii) f(nm) = f(n)f(m)$$

Proposed by Neculai Stanciu-Romania

U.1633 If ABC is a triangle with usual notations and $x, y, z > 0$ then prove that:

$$\frac{y+z}{x}r_a^2 + \frac{z+x}{y}r_b^2 + \frac{x+y}{z}r_c^2 \ge 2s^2$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

U.1634 If ABC is a triangle with usual notations and $x, y, z > 0$ then prove that:

$$\frac{y+z}{x}h_a^2 + \frac{z+x}{y}h_b^2 + \frac{x+y}{z}h_c^2 \ge \frac{4s^2r}{R}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

U.1635 If ABC is a nonisosceles triangle (with usual notations), then prove that:

$$\sum_{cyc}\frac{a^8}{(b+c)(a-b)^2(a-c)^2}>144\sqrt{3}r^3$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu-Romania

U.1636 Prove that in any triangle ABC holds:

$$\frac{bc}{b+c}+\frac{ca}{c+a}+\frac{ab}{a+b}\geq\frac{27Rr}{2s}$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu-Romania

U.1637 Compute:

$$\Omega=\int_0^{2\pi}\frac{\sin x\cdot\cos 2x}{(1+\sin^2 x)(1+\sin^2 2x)}dx$$

Proposed by Neculai Stanciu-Romania

U.1638 If $(x_n)_{n\geq 0}$ verify $x_0=x_1=1$ and $x_n=x_{n-1}-x_{n-2}$, then prove that $(x_n)_{n\geq 0}$ is periodic and find the general term of sequence.

Proposed by Neculai Stanciu-Romania

U.1639 Let $A_1A_2\ldots a_n, n\geq 3$ be a regular polygon, M a point on incircle and N a point on circumcircle of the polygon of the polygon. Prove that:

$$\sum_{k=1}^{n}\frac{MA_k^4}{NA_k^2}\geq\frac{1}{2}\left(3+\cos\frac{2\pi}{n}\right)\sum_{k=1}^{n}MA_k^2$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu-Romania

U.1670 Determine all right angled triangles with the area A, the perimeter P and the sides a,b,c natural numbers with $\gcd(a,b,c)=1$ such that $\frac{P^2}{A}$ is also natural number.

Proposed by Neculai Stanciu-Romania

U.1671 Solve in $\mathbb{Z}^3$ the following system of equations:

$$\begin{cases}xy^2+xy-x-y-1=0\\ xz-x^2-x-z=0\end{cases}$$

Proposed by Neculai Stanciu-Romania

U.1672 Evaluate:

$$\Omega=\int_0^{\frac{\pi}{2}}\frac{\cos y\,(2\cot y+\cot y\cdot\csc^2 y)\sin^3 y}{\sqrt{1-\sin^8 y}}dy$$

Proposed by Naren Bhandari- Nepal

U.1673 Prove that:

$$\sum_{n=0}^{\infty}(-1)^n\binom{2n}{n}\frac{H_{\frac{n}{2}}-H_{\frac{n-1}{2}}}{4^n(2n+1)}=8G+\pi\log(1+\sqrt{2})-4Li_2\left(\frac{1-i}{\sqrt{2}}\right)i+2i\left(\left(2Li_2(-1)^{\frac{3}{4}}\right)+\pi\tan^{-1}(\sqrt[4]{-i})\right)$$

where $H_n=\int_0^1\frac{1-x^n}{1-x}dx$, $Li_2(x)$ −is dilogarithm function and i is imaginary unit, G −Catalan's constant.

Proposed by Naren Bhandari- Nepal

U.1674 Find:

$$\Omega(m)=\int_0^{\infty}\int_0^{\infty}\frac{\log^m(xy)}{(1+x^2)(1+y^2)}dxdy\,,m\in\mathbb{R}$$

Proposed by Togrul Ehmedov-Azerbaijan

U.1675

$$\Omega(m,n,r)=\int_0^1\frac{mr\tan^{-1}\frac{1}{n\sqrt{1+m^2x^2}}}{(1+(1+r^2)m^2x^2)\sqrt{1+m^2x^2}}dx$$

Prove that: $\Omega(m,n,r)+\Omega\left(\frac{1}{n},\frac{1}{m},\frac{1}{r}\right)=\tan^{-1}\frac{\sqrt{1+r^2}}{nr}\tan^{-1}m\sqrt{1+r^2}$

Proposed by Hikmat Mammadov-Azerbaijan

U.1676 Prove that:

$$\int_0^{\infty}\left(\frac{1}{\log\left(1+\frac{1}{z}\right)}-z-\frac{1}{2}\right)^2dz=\frac{\zeta(3)}{2\pi^2}-\frac{1}{24}$$

Proposed by Hikmat Mammadov-Azerbaijan

U.1677 Prove that:

$$\sum_{n=1}^{\infty}\frac{\overline{H}_{2n}}{n4^n}\binom{2n}{n}=\frac{\pi^2}{12}$$

where H_n-n^{th} skew-harmonic number $\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}$.

Proposed by Naren Bhandari-Nepal

U.1678 Prove that:

$$\lim_{n\to\infty}\left(n\zeta(q+1)-\sum_{k=1}^{n-1}\left(\frac{n-k}{k^{q+1}}\right)\right)=\zeta(q)$$

where $\zeta(q)$ is zeta function.

Proposed by Syed Shahabudeen-India

U.1679 Find:

$$\Omega=\int_0^1 \frac{Li_2(x)\cdot\log(1-x)}{x}dx+\int_0^1 \frac{Li_2(x)\cdot\log(x)}{1-x}dx$$

Proposed by Togrul Ehmedov-Azerbaijan

U.1680 Find:

$$\Omega=\int_0^1 (\tan^{-1}x)^2\,dx$$

Proposed by Togrul Ehmedov-Azerbaijan

U.1681 If $A_n=\sin\left(\frac{1}{n^3}\right)+\sin\left(\frac{2}{n^3}\right)+\cdots+\sin\left(\frac{n}{n^3}\right)$ then prove:

$$\lim_{n\to\infty}A_n=0$$

Proposed by Akerele Olofin-Nigeria

U.1682 Prove that:

$$\int_0^1 \frac{(\tan^{-1}x)^4}{1+x}dx=\frac{\pi^4\log 2}{512}+\frac{\pi^2}{32}G-\frac{63\pi^2}{512}\zeta(3)+\frac{1395}{512}\zeta(5)+\frac{\pi}{2048}\left[\psi^{(3)}\left(\frac{3}{4}\right)-\psi^{(3)}\left(\frac{1}{4}\right)\right]$$

where G –Catalan's constant, $\zeta(s)$ –zeta function, $\psi^{(n)}(z)$ –polygamma function.

Proposed by Ankush Kumar Parcha-India

U.1683 Prove that:

$$\int_{-\infty}^{\infty}\frac{\cos m\tan^{-1}(ax)}{(1+x^2)(1+(ax)^2)^{\frac{m}{2}}}dx=\frac{\pi}{(1+a)^m},a,m>0$$

Proposed by Hikmat Mammadov-Azerbaijan

U.1684 Prove that:

$$\int_0^1\int_0^1 \sqrt{x^2+y^2}\tan^{-1}\left(\sqrt{x^2+y^2}\right)dxdy=\frac{\pi}{6}\sqrt{2}+\frac{\pi}{6}\log\left(1+\sqrt{2}\right)-\frac{2}{3}$$

Proposed by Asmat Qatea-Afghanistan

U.1685 If $\begin{cases} p+q-\sqrt{\frac{p+q}{p-q}}=\frac{12}{p-q} \\ pq=15 \end{cases}$ prove that set solution:

$$(p,q)=\left[(5,3),\left(-\sqrt{\frac{3\sqrt{109}+9}{2}},-\sqrt{\frac{3\sqrt{109}-9}{2}}\right]\right.$$

Proposed by Hikmat Mammadov-Azerbaijan

U.1686 Find: $\Omega=\int_1^2\frac{\tan^{-1}(x-1)\log x}{x}dx$

Proposed by Ose Favour-Nigeria

U.1687 Find:

$$\Omega=\int_a^b\frac{cx+f(x-a)}{c(a+b)+f(x-a)+f(b-x)}dx\,,c\geq 1,b>a$$

$f:[a,b]\to\mathbb{R}$ continuous function, $c(a+b)+f(x-a)+f(b-x)\neq 0,\forall x\in[a,b]$

Proposed by Radu Diaconu-Romania

U.1688 $A_1A_2\dots A_n$ −convex polygon, $n\in\mathbb{N},n\geq 3$. Prove that:

$$\left(\sum_{k=1}^n\mu(A_k)+\frac{1}{\prod_{k=1}^n\mu(A_k)}\right)\prod_{k=1}^n\mu(A_k)\geq(n+1)\prod_{k=1}^n\left((n-2)\pi+(1-n)\mu(A_k)\right)$$

Proposed by Radu Diaconu-Romania

U.1689 In acute $\Delta ABC,H$ −orthocenter, ΔDEF −pedal triangle of incenter, R_1 −circumradii of ΔBHC. Prove that:

$$\mu(A)\cdot EF^2+\mu(B)\cdot FD^2+\mu(C)\cdot DE^2\leq\frac{3\pi}{4}\cdot R_1^2$$

Proposed by Radu Diaconu-Romania

U.1690 In acute $\Delta ABC,H$ −orthocenter, the following relationship holds:

$$4R^2\sum_{cyc}\frac{AH}{\sin A}=\prod_{cyc}\frac{AH}{\sin A}+\frac{32R^3}{\sin 2A+\sin 2B+\sin 2C}$$

Proposed by Radu Diaconu-Romania

U.1691 In acute ΔABC the following relationship holds:

$$\sum_{cyc}\left(\frac{\cos A}{1+3\cos B\cos C}+\frac{\sin A}{1+3\sin B\sin C}\right)\geq\frac{\left(\frac{R+r}{R}\right)^2}{\frac{17}{8}+\frac{r}{R}}+\frac{\left(\frac{s}{R}\right)^2}{\frac{27\sqrt{3}}{8}+\frac{s}{R}}$$

Proposed by Radu Diaconu-Romania

U.1692 In acute $\Delta ABC, m_b \perp m_c$ then:

$$\frac{\sin B\sin C}{\sin^2 A}\left(\sum_{cyc}\mu(A)\sqrt[3]{\cos A}\right)<\frac{15\sqrt{3}}{4}$$

Proposed by Radu Diaconu-Romania

U.1693 In right ΔABC the following relationship holds:

$$\frac{\mu(A)}{\sqrt{s-a}}+\frac{\mu(B)}{\sqrt{s-b}}+\frac{\mu(C)}{\sqrt{s-c}}\ge \pi\sqrt{\frac{3}{2R+r}}$$

Proposed by Radu Diaconu-Romania

U.1694 In $\Delta ABC, B'\in(AC), B'A=B'C, C'\in(AB), C'A=C'B, BB'\perp CC', r_1, r_2, r_3$ −inradii in $\Delta BGC, \Delta B'GC, \Delta C'GB, R_1$ −circumradii in ΔBGC. Prove that:

$$6(r_1+r_2+r_3)+3(s+R_1)=5(m_b+m_c)$$

Proposed by Radu Diaconu-Romania

U.1695 In$\Delta ABC, H$ −orthocenter, $r_a-r=3R$. Prove that:

$$AH\ge \min\{b,c\}$$

Proposed by Radu Diaconu-Romania

U.1696 In ΔABC holds:

$$\begin{vmatrix} x & 0 & -1 & 1 & 0\\ 1 & x & -1 & 1 & 0\\ 1 & 0 & x-1 & 0 & 1\\ 0 & 1 & -1 & x & 1\\ 0 & 1 & -1 & 0 & x \end{vmatrix}\ge 0,\text{ where } x=-8\prod_{cyc}\sin\frac{A}{2}$$

Proposed by Radu Diaconu-Romania

U.1697 Find without softs: $\Omega(n)=\int_{5-2\sqrt{6}}^{5+2\sqrt{6}}\frac{\log(2x^n)}{1+x^2}dx\,; n\ge 1$

Proposed by Radu Diaconu-Romania

U.1698 In acute $\Delta ABC, a\le b\le c, D\in(BC), \delta$ −circumradii of $\Delta ABD, \rho$ −inradii of $\Delta ACD, O$ −circumradius,I −incenter. Prove that:

$$\frac{bc}{\mu(A)}+\frac{ca}{\mu(B)}+\frac{ab}{\mu(C)}\ge\frac{6\rho(OI^2+36r^2)}{\pi\delta}$$

Proposed by Radu Diaconu-Romania

U.1699 If $VABC$ –tetrahedron, then:

$$\frac{([VAB]+[VBC]+[VCA])\cdot[ABC]}{(VA^2+VB^2+VC^2)(AB^2+BC^2+CA^2)}\le\frac{1}{12}$$

Proposed by Radu Diaconu-Romania

U.1700 In $\Delta ABC, c_a, c_b, c_c$ –cevians holds:

$$\max\{1+c_ar_a, +1c_br_b, 1+c_cr_c\}\ge F\cdot\sqrt[3]{\frac{2s}{R}+8\prod_{cyc}\sin\frac{A}{2}}$$

Proposed by Radu Diaconu-Romania

U.1701 In ΔABC the following relationship holds:

$$\frac{\cot\frac{A}{2}}{ar_a\mu(A)}+\frac{\cot\frac{B}{2}}{br_b\mu(B)}+\frac{\cot\frac{C}{2}}{cr_c\mu(C)}>\frac{9\sqrt{3}\cdot m_a}{2sr\pi\cdot\max\{b,c\}}$$

Proposed by Radu Diaconu-Romania

U.1702 If N –Nagel's point in ΔABC then: $\frac{NA}{\sin A}+\frac{NB}{\sin B}+\frac{NC}{\sin C}>\frac{am_a+bm_b+cm_c}{s}$

Proposed by Radu Diaconu-Romania

U.1703 For any complex number $n, \Re(n)>0$, prove that:

$$\int_{-\infty}^{\infty}e^{-x}j_n(e^{-x})dx=\frac{\sqrt{\pi}\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(1+\frac{n}{2}\right)},\qquad \text{where } j_n(x)=x^n\sum_{n=0}^{\infty}\frac{(-1)^k\left(\frac{x^2}{2}\right)^k}{k!\,(2k+2n+1)!!}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.1704 If $A_1A_2\dots A_n$ –convex polygon $n\in\mathbb{N}, n\ge 3$ then:

$$\sum_{k=1}^{n}\frac{(n-2)\pi-\mu(A_i)}{(n-2)\pi+\mu(A_i)}\ge\frac{n(n-1)}{n+1}$$

Proposed by Radu Diaconu-Romania

U.1705 If we have the function $S(n)=\frac{1}{4}\sin\left(\frac{n\pi}{2}\right)+\frac{4}{9}\sin\left(\frac{2\pi n}{3}\right)+\frac{9}{16}\sin\left(\frac{3\pi n}{4}\right)$ then prove the sum:

$$\sum_{n=1}^{\infty}\frac{S(-n)}{n^3S'(n)}=\frac{3318639892950782842891}{4579116479913033682790 4}\pi^2, \text{here } S'(n)=\frac{\partial S(n)}{\partial n}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.1706 $ABCD$ −parallelogram, $AB + AD \geq n > 0$. Prove that:

$$\max\{AC^2, BD^2\} \geq \frac{n^2}{2}$$

Proposed by Radu Diaconu-Romania

U.1707 Prove the summation:

$$\sum_{n=1}^{\infty} \frac{\left(\sqrt{1}\sin(n\pi) + \sqrt{3}\sin\left(\frac{n\pi}{3}\right) + \sqrt{5}\sin\left(\frac{n\pi}{5}\right)\right)^2}{n^4} = \frac{4\pi^4(29\sqrt{15} + 4)}{10125} + \frac{2\pi^4}{45}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.1708 $\psi \leq 0, m, n, p \geq 1, m, n, p \in \mathbb{N}$. In acute ΔABC holds:

$$\frac{A^n}{h_a^m \cos^p A + \psi} + \frac{B^n}{h_b^m \cos^p B + \psi} + \frac{C^n}{h_c^m \cos^p C + \psi} \geq \frac{6^p \cdot \pi^n}{3^{m+n+p-1} \cdot r^m}$$

Proposed by Radu Diaconu-Romania

U.1709 Let ΔDEF be the circumcevian triangle of circumcentre in ΔABC. Prove that:

$$\left(\sqrt{\frac{a\cos A}{AD}} + \sqrt{\frac{b\cos B}{BE}} + \sqrt{\frac{c\cos C}{CF}}\right)^2 \leq \frac{s}{r}$$

Proposed by Radu Diaconu-Romania

U.1710 In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{b^2 + c^2}{s + a} \cdot \sum_{cyc} \frac{\mu^2(B) + \mu^2(C)}{s + a} \geq \left(\frac{4\pi}{5}\right)^2$$

Proposed by Radu Diaconu-Romania

U.1711 Find:

$$\Omega = \int_0^{\infty} \frac{x\tan^{-1} x}{(x + 1)(x^2 + 1)} dx$$

Proposed by Vasile Mircea Popa-Romania

U.1712 In ΔABC the following relationship holds:

$$243r^4 \leq m_a w_a^3 + m_b w_b^3 + m_c w_c^3 \leq \frac{27}{16}\max\{a^4, b^4, c^4\}$$

Proposed by Radu Diaconu-Romania

U.1713 If we define the function $S(n)$ for $n > 0$,

$S(n) = \int_{-\infty}^{\infty} \sin(\pi x^2) \sin^2\left(\frac{1}{2}\pi x(n+x)\right) dx$ then prove the sum:

$$\sum_{n=1}^{\infty} \frac{S(n)}{n^2} = \frac{\pi^2}{384}\left(1 + 16\sqrt{2} - 3\sqrt{2-\sqrt{2}}\right)$$

Proposed by Srinivasa Raghava-AIRMC-India

U.1714 Prove the summation:

$$\sum_{n=-\infty}^{\infty} \frac{\cos\left(\frac{\pi}{3}(4n-2)\right)}{(4n-2)^2} e^{-\frac{1}{3}i\pi n} = \frac{\pi^2}{48}(-1)^{\frac{5}{6}}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.1715 Find:

$$\Omega = \int_0^1 \int_0^1 (x^2 + 2xy + x) \log\left(1 + \frac{1}{x+y}\right) dxdy$$

Proposed by Asmat Qatea-Afghanistan

U.1716

For $S(n) = \sum_{k=1}^{2^{n+1}-1} \log_2\left(1 + \frac{1}{\sqrt{k^2+k+1}}\right)$ find: $\Omega = \lim_{n\to\infty}\left(\sum_{k=1}^{n} \frac{1}{k\sqrt{k+1} + (k+1)\sqrt{k}}\right)^{\frac{S_n}{\sqrt{n}}}$

Proposed by Florică Anastase-Romania

U.1717

For $\Omega_1 = \int_0^1 \log x \sqrt{\frac{x+1}{x-x^3}} dx$, $\Omega_2 = \int_1^{\infty} \frac{\log x}{x\sqrt{x-1}} dx$, $\Omega_3 = \int_0^1 \log\left(\frac{1}{1-x^2}\right)\sqrt{\frac{1+x}{x-x^3}} dx$

$$\Omega_4 = \int_1^{\infty} \frac{\log(1+x)}{x\sqrt{x-1}} dx \text{ prove: } \Omega_1 + \Omega_2 - \Omega_3 - \Omega_4 = -\pi \log 16$$

Proposed by Ankush Kumar Parcha-India

U.1718 Prove that:

$$\sum_{k=1}^{\infty} \frac{(\sqrt{2})^{-k}}{k^3} \cos\left(\frac{k\pi}{4}\right) = \frac{35\zeta(3)}{64} + \frac{\log^3 2}{48} - \frac{5}{192}\pi^2 \log 2$$

Proposed by Fao Ler-Iraq

U.1719 If $a, b > 0$ and $1 - a < c < 1 + \min\{a, b\}$ find:

$$\Omega = \int_0^\infty \frac{\cos(x^a) - e^{-x^b}}{x^{1+c}} dx$$

Proposed by Hikmat Mammadov-Azerbaijan

U.1720 If $k > 1$ compare it: $\Omega_1 = \int_0^\infty \sin(z^k)\, dz$ and $\Omega_2 = \int_0^\infty \cos(z^k)\, dz$

Proposed by Hikmat Mammadov-Azerbaijan

U.1721 For a constant, $0 < a < b$ find Ω such that:

$$e^{\Omega} = \sum_{k=-\infty}^{\infty} \left(\frac{a}{b}\right)^{|k|} \sum_{n=0}^{\infty} \frac{(ix)^n k^n}{n!}, x \in \mathbb{R}$$

Proposed by Tobi Joshua-Nigeria

U.1722 Find:

$$\Omega = \int_0^1 \int_0^1 \frac{x \cdot (\tan^{-1} x)^2}{1 + xy} dxdy$$

Proposed by Togrul Ehmedov-Azerbaijan

U.1723 If $a, b > 0$ then:

$$\int_0^\infty \sin\left(a^2x^2 - \frac{b^2}{x^2}\right) dx = \frac{\sqrt{2\pi}}{4a} e^{-2ab}$$

Proposed by Abdul Mukhtar-Nigeria

U.1724 Find $m, n \in \mathbb{Z}$ such that: $\int_0^\infty \frac{x^m}{(x^n+1)^2} dx = \pi$

Proposed by Abdul Mukhtar-Nigeria

U.1725 Find:

$$\Omega(a) = \int_0^\infty \frac{1}{(x^2 + x + 1)(1 + a^2x^2)} dx, a \in \mathbb{R}$$

Proposed by Vasile Mircea Popa-Romania

U.1726 Compute:

$$a)\ \lim_{n\to\infty} \left(\frac{(n+1)^2}{\sqrt[n+1]{a \cdot (n+1)!}} - \frac{n^2}{\sqrt[n]{a \cdot n!}}\right), a > 0$$

$b)$ If $\lim\limits_{n\to\infty} a_n = a, a > 0,$ then find $\lim\limits_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{a_n\cdot(n+1)!}} - \frac{n^2}{\sqrt[n]{a_n\cdot n!}}\right)$

Proposed by Neculai Stanciu-Romania

U.1727 If $n \in \mathbb{N}$, then prove that:

$$\sum_{k|n}\sum_{K'|k} d(k') = \sum_{k|n} d(k)d\left(\frac{k}{n}\right)$$

Proposed by Angad Singh-India

U.1728 Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{(n^2-2)(n-1)!}\left(\sum_{k=2}^{n}(k^3-1)k!\right)\right)^{(n^2-2)(n-1)!}$$

Proposed by Daniel Sitaru-Romania

U.1729 If $0 < a \le b < \frac{\pi}{2}$ then:

$$\int_a^b\int_a^b \sqrt{4-(\sin x+\sin y)^2}\,dxdy \ge 2(b-a)(\cos a-\cos b)$$

Proposed by Daniel Sitaru-Romania

U.1730 Find:

$$\Omega = \lim_{n\to\infty} n\left(3^{\sqrt[n]{\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}}} - 1\right)$$

Proposed by Daniel Sitaru-Romania

U.1731 Find:

$$\Omega = \lim_{n\to\infty}\frac{(1+H_1)(1+H_2)\cdot\ldots\cdot(1+H_n)}{2^n\cdot H_n(1+H_1H_2\cdot\ldots\cdot H_n)}$$

Proposed by Daniel Sitaru-Romania

U.1732 If $0 < a \le b$ then:

$$\int_{\sqrt{ab}}^{\frac{a+b}{2}} e^{-x^2}\,dx \le \tan^{-1}\left(\frac{a+b}{2}\right) - \tan^{-1}\left(\sqrt{ab}\right)$$

Proposed by Daniel Sitaru-Romania

U.1733 If $a, b > 0$ then:

$$\frac{4}{a+b} \le \int_0^1 \frac{dx}{xa+(1-x)b} + \int_1^\infty \frac{dx}{(x+a)(x+b)} \le \frac{2}{\sqrt{ab}}$$

Proposed by Daniel Sitaru-Romania

U.1734 If $0 < a \le b, n \in \mathbb{N}, n \ge 2$ then find:

$$\Omega(a,b,n) = \int_a^b \left(\frac{1}{x^{n+1}} \int_0^x y^n \sin\left(y + \frac{(n+1)\pi}{2}\right) dy\right) dx$$

Proposed by Daniel Sitaru-Romania

U.1735 Find:

$$\Omega = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left| \left(\frac{\sin x}{x}\right)^{(k)} \right|, x \in \left(0, \frac{\pi}{2}\right); (*) - k^{th} \text{ derivative.}$$

Proposed by Daniel Sitaru-Romania

U.1736 Prove that:

$$\int_1^\infty \int_1^\infty \frac{\sqrt{x}+\sqrt{y}}{\left(1+\sqrt{xy}\right)\left(1+xy\left(1+xy(1+xy)\right)\right)} dydx = \pi\left(\sqrt{2}-\frac{3}{2}\right) + \log\left(\frac{1}{8}\left(2\sqrt{2}+3\right)^{\sqrt{2}}\right)$$

Proposed by Srinivasa Raghava-AIRMC-India

U.1737 If $a, m > 0$ then prove:

$$\int_{-\infty}^{\infty} \frac{\cos m \tan^{-1}(ax)}{(1+x^2)(1+(ax)^2)^{\frac{m}{2}}} dx = \frac{\pi}{(1+a)^m}$$

Proposed by Hikmat Mammadov –Azerbaijan

U.1738 Find a closed form: $\Omega = \sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n^3 \sinh(\pi n)}$

Proposed by Asmat Qatea-Afghanistan

U.1739 Prove that:

$$\int_0^1 \int_0^1 \frac{x^2 y^3 \log x \log y}{(1-x^2y^2)(1-x^2)(1-y^2)} dxdy =$$

$$= \frac{91\pi^4}{11520} + \frac{21}{32}\zeta(3) - \frac{7}{8}\zeta(3)\log 2 - Li_4\left(\frac{1}{2}\right) + \frac{\pi^2}{24}\log^2 2 - \frac{\pi^2}{16}\log 2 - \frac{\log^4 2}{24}$$

Proposed by Narendra Bhandari-Nepal

U.1740 Prove that:

$$\int_0^{\frac{1}{\sqrt{3}}} \frac{\tan^{-1}\left(\frac{1}{\sqrt{1-2x^2}}\right)}{1+x^2}dx = \frac{13\pi^2}{288}$$

Proposed by Djahel Hamza-Algerie

U.1741 Find:

$$\Omega = \int_0^{\infty} \frac{4\log(1+x^2)\log x}{x^5+x}dx$$

Proposed by Daniel Immarube-Nigeria

U.1742 Prove that: $\int_0^{\infty} \frac{\log(1+y)}{y(1+y^2)}dy = \frac{5}{8}\zeta(2)$

Proposed by Ankush Kumar Parcha-India

U.1743 Find:

$$\Omega = \int_0^1 \frac{x\log^2(1+x)}{(x^2+1)^2}dx$$

Proposed by Togrul Ehmedov-Azerbaijan

U.1744 Prove that:

$$\int_0^{\frac{1}{2}} \frac{\tan^{-1}\left(\sqrt{1-2x^2}\right)}{1+2x^2}dx = \frac{1}{2\sqrt{2}}\left(\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)\right)^2 + \frac{\pi}{4\sqrt{2}}\tan^{-1}\left(\frac{1}{2\sqrt{2}}\right)$$

Proposed by Djahel Hamza-Algerie

U.1745 Prove that:

$$\int_0^1 \frac{(1+x)\log^2 x}{(1-x)^3}\left(Li_2(x) - \frac{\pi^2}{6}\right)dx = 2\zeta(2) - 10\zeta(10)$$

Proposed by Narendra Bhandari-Nepal

U.1746 Let $d(n)$ denotes the number of digits of n and $\lambda(k) = \frac{(d(n))^k}{n}$, where n ranges over positive integers. Prove that:

$$\sum_{k=1}^{\infty} \frac{\max\{\lambda(k)\}}{3^{3k}} \cdot \frac{(3^k \cdot 6 - 1)}{k^2} = \frac{1}{100}\left(\frac{\pi^2}{3} - \log^2 3\right)$$

Proposed by Narendra Bhandari-Nepal

U.1747 Let $H_n -$ be n^{th} harmonic number, then prove:

$$\sum_{n=1}^{\infty}\binom{2n}{n}\frac{H_{2n}-H_n}{4^n(2n-1)^2}=\frac{3\pi}{2}\log 2-2G-\pi+2$$

where G −Catalan's constant, defined as $G=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^2}$.

Proposed by Narendra Bhandari-Nepal

U.1748 Determine the values of m and n if:

$$\lim_{x\to\infty} a\left(2-x\sin\frac{m}{x}\right)\left(\Gamma\left(\frac{1}{x}\right)\right)^n=1$$

Proposed by Jalil Hajimir-Canada

U.1749 Prove without any software:

$$4e\left|1-\int_0^1 e^{x^2}\,dx\int_0^1 e^{-x^2}\,dx\right|<(e-1)^2$$

Proposed by Daniel Sitaru-Romania

U.1750 Prove that:

$$\int_0^{\frac{\pi}{4}}\frac{4\log(\cot x)}{\cos(2x+2022\pi)}\,dx=3\zeta(2)$$

Proposed by Muhammad Afzal-Pakistan

U.1751 Prove the identity for any (a,n) in real number:

$$(1+a)\cdot a^{[n]}=a\cdot a^{2\left[\frac{n}{2}\right]}+a^{2\left[\frac{n+1}{2}\right]}$$

Proposed by Asmat Qatea-Afghanistan

U.1752 Prove the identity for any (a,n) in real number:

$$\left[\frac{n}{2}\right]\cdot\left[\frac{n+1}{2}\right]=\frac{1}{4}\left([n]^2+2\left[\frac{n}{2}\right]-[n]\right)$$

Proposed by Asmat Qatea-Afghanistan

U.1753 Find:

$$\Omega=\left(\sum_{k=0}^{99}\binom{500}{5k}\cdot\binom{500}{5k+2}\right)\left(\sum_{k=1}^{100}\binom{1000}{10k-8}\right)$$

Proposed by Adrian Popa-Romania

U.1754 Prove that:

$$\int_0^{\frac{\pi}{2}} \frac{e^{\cos 2x} \cdot \sin(x + sin\, 2x)}{\sin x} dx = \frac{\pi e}{2}$$

Proposed by Asmat Qatea-Afghanistan

U.1755

$$\int_{-\infty}^{\infty} \frac{\sin\left(\frac{\pi x^2}{2} + \frac{\pi}{8}\right) \sin\left(\frac{\pi x^2}{2} - \frac{\pi}{8}\right) \cos^2(\pi x)\, \Gamma\left(x + \frac{1}{2}\right)}{\cosh(\pi x)\, \Gamma(1 + x)} dx = \beta \int_0^1 \frac{\sin(\pi x^2)\sqrt{1 + \csc(\pi x)}}{\cosh(\pi x)} dx$$

Find β.

Proposed by Balendran Sujeethan-Sri Lanka

U.1756 Prove that:

$$\int_0^{\frac{\pi}{2}} x^4\left(\sqrt{\tan x} + \sqrt{\cot x}\right)dx =$$

$$= \sqrt{2}\pi \left[\frac{3}{8}\zeta(3)\log 2 + \frac{79}{3840}\pi^4 + \frac{\log^4 2}{16} - \frac{5}{32}\pi^2 \log^2 2\right]$$

Proposed by Balendran Sujeethan-Sri Lanka

U.1757 Prove that:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \psi^{(0)}\left(\frac{n}{3}\right) = \gamma \log 2 + \log 2 \log 3 + \frac{1}{2}\log^2 2 + \frac{4}{3}\zeta(2) + \frac{Li_2(-3)}{2}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \psi^{(0)}\left(\frac{n+1}{3}\right) = \gamma \log 2 + \frac{9}{4}\Phi\left(-3,2,\frac{3}{2}\right) + \frac{1}{4}Li_2\left(-\frac{1}{3}\right) - 3 -$$

$$-\frac{23}{36}\zeta(2) + \frac{1}{2}\log^2 2 + \frac{1}{8}\log^2 3 + \frac{\pi}{2\sqrt{3}}\log 2 + \log 2 \log 3 + \frac{1}{18}\psi^{(1)}\left(\frac{1}{6}\right) +$$

$$+\frac{1}{36}\psi^{(1)}\left(\frac{1}{3}\right) - \frac{1}{18}\psi^{(1)}\left(\frac{2}{3}\right) - \frac{1}{36}\psi^{(1)}\left(\frac{5}{6}\right)$$

Proposed by Surjeet Singhania-India, Narendra Bhandari-Nepal

U.1758 Prove that:

$$\int_{-\infty}^{\infty} \frac{(ix - a)^{-n}}{(-1)^n} e^{-ibx} dx = \frac{2\pi b^n}{be^{ab}\Gamma(n)}, b > 0, \Re(n) > 0, \Re(a) > 0$$

where i is complex number and $\Gamma(x)$ –is gamma function.

Proposed by Tobi Joshua-Nigeria

U.1759 Solve for real numbers:

$$4\tan x+\frac{\sin(5x)}{\cos^5 x}=0$$

Proposed by Togrul Ehmedov-Azerbaijan

U.1760 Find:

$$\Omega=\int_0^1 \frac{x\cdot Li_2(1-x)}{1+x^2}dx$$

Proposed by Togrul Ehmedov-Azerbaijan

U.1761 Prove that:

$$\frac{d^2y}{dx^2}-3\left(\frac{dy}{dx}\right)-4y=\tan x\log(cos\,x)$$

Proposed by Daniel Immarube-Nigeria

U.1762 If

$$\Omega=\int_0^1\underbrace{\cdots}_{2023}\int_0^1\frac{(-1)}{\sum_{i=1}^{2023}\log(x_i)}\prod_{i=1}^{2023}dx_i$$

Then find the value of Ω^{-1}.

Proposed by Syed Shahabudeen-India

U.1763 Prove that: $[\pi^6+e^6+\pi^5+e^5+\pi^4+e^4+\pi^3+e^3]=2022, [*]-\text{GIF}.$

Proposed by Srinivasa Raghava-India

U.1764 Find:

$$\Omega=\frac{1}{\pi}\sum_{k=0}^{\infty}\left(\frac{3^{k+1}-1}{2^{k+1}-1}\right)\frac{\eta(k+2)}{2^{k+1}}$$

where $\eta(s)$ —is the Diriclet's eta function. Let $z=f(x,y), u=\log(x^2+y^2), v=xy$. Determine: $\delta=\frac{2(x^2-y^2)}{x}\cdot\frac{\partial x}{\partial u}$ and find:

$$\Phi(y)=\int_0^{\infty}\frac{1}{\delta x}(\log^2\delta-4y\log y)\,dx$$

Proposed by Abdul Hafeez Ayinde-Nigeria

U.1765 Find a closed form:

$$\Omega = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{n^2(m^2+n^2)m^2}$$

Proposed by Surjeet Singhania-India

U.1766 Let $u_0 = 2, u_1 = 4, u_n + u_{n+2} = 4u_{n+1}, n \in \mathbb{N}$. Find: $\Omega = \left(\sum_{k=1}^{n} u_k\right)(\,mod\ 2)$

Proposed by Surjeet Singhania-India

U.1767 For $|x| \leq 1$, let $T(x) = \sum_{n=1}^{\infty}\frac{\left((-1)^{\frac{1}{2}n(n+1)}+1\right)x^n}{n^2+n}$ then prove the relation:

$$\int_0^1\left(x+\frac{1}{x}\right)\left(T(-x)+T(x)\right)dx = \frac{\pi^2}{24} - 1 + \log 2$$

Proposed by Srinivasa Raghava-AIRMC-India

U.1768 For $n \geq 1$ let $\int_0^{\infty}\frac{e^{-\pi x}\cos\left(\frac{\pi x}{n}-\pi n x\right)}{\sqrt{x}}dx = f(n)\int_0^{\infty}\frac{e^{-\pi x}\cos\left(\pi n x+\frac{\pi x}{n}\right)}{\sqrt{x}}dx$ then prove:

$$\frac{f(\sqrt{\varphi})}{f(\varphi)} = \frac{\sqrt{\frac{\left(\sqrt{\sqrt{5}+1}+2\right)\left(\sqrt{5}+2\sqrt{3(\sqrt{5}+3)}+1\right)}{2\sqrt{2}+\sqrt{5}+1}}}{\sqrt{3}\sqrt[4]{\sqrt{5}+1}}, \varphi - \text{golden ratio.}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.1769 Show that:

$$\int_0^{\frac{\pi}{4}}\int_0^1\int_0^1\frac{a^2+(b\sin x)^2}{b^2+(a\cos x)^2}\tan x\, dadbdx =$$

$$= \frac{1}{12} - \frac{5\pi}{12} + \frac{\log 3}{6} + \frac{5\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)}{6\sqrt{2}} + \frac{17\tan^{-1}(\sqrt{2})}{12\sqrt{2}}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.1770 Evaluate the integral:

$$\int_0^1\frac{Li_{\frac{1}{3}}\left(\sqrt[3]{1-x}\right)\log\left(\sqrt[3]{1-x}\right)}{\sqrt[3]{1-x}}dx$$

$Li_{\frac{1}{3}}(x)$ – is Poly-Logarithm function.

Proposed by Srinivasa Raghava-AIRMC-India

U.1771 Prove that:

$$\frac{26\zeta(3)-80}{79\zeta(3)+3}\leq\frac{\sin(e^{-x})}{e^{\cos x}+1}\leq\frac{1}{2+2\log 2}$$

$\zeta(3)$ is Apery's constant.

Proposed by Srinivasa Raghava-AIRMC-India

U.1772 Prove that:

$$\sum_{n=0}^{\infty}e^{-\pi n^2}(1-4\pi n2\,)(-1)^{\frac{1}{2}n(n+1)}=\frac{3}{2}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.1773 Solve the system:

$$\begin{cases}b\sqrt{a}+a\sqrt{b}=a+b\\ \sqrt{a}+\sqrt{b}=ab\end{cases}$$

Find all the real solutions other than $a=0,b=0$.

Proposed by Srinivasa Raghava-AIRMC-India

U.1774 Prove the product:

$$\prod_{n=1}^{\infty}\frac{\left(1+\frac{1}{(3n)^2}\right)^{(3n)^2}}{\left(1-\frac{1}{(4n-2)^2}\right)^{(4n-2)^2}}=\sqrt{\frac{3}{2}e^{\alpha-\frac{1}{2}}}$$

where $\alpha=\frac{4G}{\pi}-\frac{47\zeta(3)}{4\pi^2}-\frac{2\pi}{3\sqrt{3}}+\frac{\phi^{(1)}\left(\frac{1}{3}\right)}{\sqrt{3}\pi}$, G –Catalan's constant, $\psi^{(1)}(t)$ –Polygamma function.

Proposed by Srinivasa Raghava-AIRMC-India

U.1775 Solve for a,b,c with $a,b,c>0$:

$$\left(\frac{2}{a}+\frac{3}{b}+\frac{5}{c}\right)^2=\left(\frac{a}{2}+\frac{b}{3}+\frac{c}{5}\right)(abc)$$

Proposed by Srinivasa Raghava-AIRMC-India

U.1776 Prove the integral relation:

$$\left|\int_{-\infty}^{\infty}e^{\frac{-\pi x(x+1)}{2}}(-1)^{\frac{1}{2}x(x+1)}\sinh(\pi x)\,dx\right|+\sqrt{2}\int_{-\infty}^{\infty}e^{-\pi x(x+1)}(-1)^{x(x+1)}\sinh(\pi x)\,dx=0$$

Proposed by Srinivasa Raghava-AIRMC-India

U.1777 If we have the equation in x

$$x+\frac{1}{x}=\left(\sqrt{1+x}+\frac{2}{\sqrt{1-x}}\right)\left(\sqrt{1-x}+\frac{\sqrt{1+x}}{2}\right)$$

then find the value of the expression: $\Omega=\sqrt{5x^5-23x^4+39x^3-33x^2+24x}$.

Proposed by Srinivasa Raghava-AIRMC-India

U.1778 If we have the product:

$$\prod_{k=1}^{n}\frac{4n^3}{4n^3+3n+1}=\Gamma(a)\Gamma(b)\Gamma(c)$$

then show that for $|x|<1$ it holds: $\sum_{m=0}^{\infty}x^m(a^m+b^m+c^m)=\frac{24x-15x^2-12}{8x^3-15x^2+12x-4}$

Proposed by Srinivasa Raghava-AIRMC-India

U.1779 If we let $f(q)=\mathcal{F}_x\left[\frac{x}{1-x}+\frac{x^4}{(1-x)(1-x^2)}\right](q)$ then show that:
$\int_0^{\pi} f(q)f(-q)\cos(q)\,dq=\frac{\pi^2}{3}$, here $\mathcal{F}_x[f](q)$ is Fourier transform.

Proposed by Srinivasa Raghava-AIRMC-India

U.1780 If we have the integral

$$\alpha=\int_0^{\infty}\int_0^{\infty}\frac{\sqrt[3]{x}}{1+\sqrt[3]{x}}e^{-\pi y\left(1+x^2+\frac{1}{x^2}\right)}dydx$$

then evaluate the expression: $\Omega=\sqrt{19683\alpha^6-94041\alpha^4+105786\alpha^2}$.

Proposed by Srinivasa Raghava-AIRMC-India

U.1781 Prove the summation:

$$2+\sum_{n=1}^{\infty}\frac{(4n-1)H_{2n-1}}{\varphi^{2n+1}}=4\varphi+\frac{1}{4}(14\varphi+22)\log\varphi$$

H_m −Harmonic number and φ −golden ration.

Proposed by Srinivasa Raghava-AIRMC-India

U.1782 Solve for real numbers: $\begin{cases}-x-y^2-z^3=2022^{12}\\ -x^{-1}-y^{-2}-z^3=2022^{-12}\\ xy^2z^3=2022^{24}\end{cases}$

Proposed by Neculai Stanciu-Romania

U.1783 Prove that:

$$\frac{x}{2x+y+z+t}+\frac{y}{x+2y+z+t}+\frac{z}{x+y+2z+t}+\frac{t}{xy+z+2t}\le\frac{4}{5};\forall x,y,z,t>0$$

Proposed by Neculai Stanciu, George Florin Șerban-Romania

U.1784 If $a>0$, then determine all real numbers x which satisfy:

$$x^2\left(a^{\sqrt{x}-x}-1\right)-\sqrt{x}+1=0$$

Proposed by Neculai Stanciu, George Florin Șerban-Romania

U.1785 If $x,y,z>0$ with $x+y+z=m$, then find: $\Omega=\min\left(\frac{1}{1+x^2}+\frac{1}{1+y^2}+\frac{1}{1+z^2}\right)$

Proposed by Marius Drăgan, Neculai Stanciu-Romania

U.1786 If $p\ge 5$ is a prime number, then determine the remainder of the division of the number p^2 at 12. ***Proposed by Neculai Stanciu-Romania***

U.1787 Determine all triplets (x,y,z) of real numbers which satisfy:

$$\begin{cases}\dfrac{x}{\sqrt{x^2-2x+4}}=\log_2(4-y)\\ \dfrac{y}{\sqrt{y^2-2y+4}}=\log_2(4-z)\\ \dfrac{z}{\sqrt{z^2-2z+4}}=\log_2(4-x)\end{cases}$$

Proposed by Neculai Stanciu, George Florin Șerban-Romania

U.1788 Solve for real numbers:

$$\begin{cases}x(x+1)=y-1\\ x^2(y+3)+2x=-1\end{cases}$$

Proposed by Neculai Stanciu-Romania

U.1789 Solve for real numbers $\{x^{2021}+x^2+x+1\}=x^{2021}$, $\{x\}$ –fractional part of x.

Proposed by Neculai Stanciu-Romania

U.1790 If $a,x\ge 0,b,c,d,k,y,z>0$ and the sequence $\{u_n(k)\}_{n\in\mathbb{N}}$ is defined recursively by $u_{n+2}=ku_{k+1}+u_n,\forall n\in\mathbb{N},u_0=x,u_1=y,z\ge y,c(u_{n+1}+u_n-x-y)>dk\max_{1\le j\le n}u_j$, then prove:

$$a)\ \sum_{j=1}^{n}\left(\frac{a(u_{n+1}+u_n-x-y)bku_j}{c(u_{n+1}+u_n-x-y)-dku_j}\right)^2\ge\frac{(an+b)^2n}{(cn-d)^2};\forall n\in\mathbb{N}^*$$

$$b)\ \sum_{j=1}^{n}\left(\frac{a(u_{n+1}+u_n-x-y)bku_j}{c(u_{n+1}+u_n-x-z)-dku_j}\right)^2 \geq \frac{(an+b)^2 n}{(cn-d)^2};\forall n\in\mathbb{N}^*$$

Proposed by D.M Bătinețu-Giurgiu, Neculai Stanciu-Romania

U.1791 If $x>0$ such that $a_1=\frac{e^x+e^{-x}}{2}\in\mathbb{N}^*, a_{n+1}=a_1a_n+\sqrt{(a_1^2-1)(a_n^2-1)}, \forall n\in\mathbb{N}^*$, then prove that all terms of the sequence $(a_n)_{n\geq 1}$ are positive integers.

Proposed by Neculai Stanciu-Romania

U.1792 Prove that in any triangle ABC holds:

$$a^2+b^2+c^2\geq \frac{4F}{\sqrt{3}}\left(\frac{w_a}{h_a}+\frac{w_b}{h_b}+\frac{w_c}{h_c}\right)>4\sqrt{3}F$$

Proposed by D.M Bătinețu-Giurgiu, Neculai Stanciu-Romania

U.1793 Determine all triangle with:

-the lengths of sides positive integers and at least one is prime number.

-the semiperimeter is positive integer and area is equal with perimeter.

Proposed by Neculai Stanciu-Romania

U.1794

$$\Omega=\int_{\sqrt{\frac{5}{7}}}^{1}\frac{\frac{\pi}{3}-\tan^{-1}\sqrt{\frac{2x^2-1}{3x^2-2}}\tan^{-1}x}{(3x^2-1)\sqrt{2x^2-1}}dx$$

Find: $A.\Omega=\frac{\pi^3}{2021}$ $B.\Omega=\frac{\pi}{2021}$ $C.\Omega=\frac{\pi^3}{2016}$ $D.\Omega=\frac{\pi}{20216}$

Proposed by Hikmat Mammadov-Azerbaijan

U.1795 If x, y and z are real numbers that are the lengths of a triangle such that

$x+y+z=1, x<2y, y<2z, z<2x$ then prove:

$$\sqrt{\frac{x}{2y-x}}+\sqrt{\frac{y}{2z-y}}+\sqrt{\frac{z}{2x-z}}\geq\frac{1}{\sqrt{3xyz}}$$

Proposed by Hikmat Mammadov-Azerbaijan

U.1796 If $x,y,z\in\mathbb{R}$ then prove that:

$$(x^2+y^2+z^2)^2-(xy+yz+zx)^2\geq\sqrt{6}(x-y)(y-z)(z-x)(x+y+z)$$

Proposed by Hikmat Mammadov-Azerbaijan

U.1797 Prove that:

$$\sum_{i=1}^{n} \frac{a_i}{(a_1 + a_2 + a_3 + \cdots + a_n) - a_i} \le \sum_{i=1}^{n} \frac{a_i^2}{(a_1^2 + a_2^2 + a_{362} + \cdots + a_n^2) - a_i^2}; a_i > 0$$

Proposed by Hikmat Mammadov-Azerbaijan

U.1798 Find:

$$\Omega = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{3^{n+1}(n+3)}$$

Proposed by Tobi Joshua-Nigeria

U.1799 Find: $\Omega = \int_0^1 \int_0^1 \log(1-x) \cdot Li_2(1-xy)dxdy$

Proposed by Togrul Ehmedov-Azerbaijan

U.1780

$$J = \int_0^1 (\tan^{-1} x)^n dx \, , n > 1$$

Show that:

$$J = \left(\frac{\pi}{4}\right)^n - n\left(\left(\frac{\pi}{4}\right)^{n-1}\left(\frac{\log 2}{2} - \frac{n-1}{n} \cdot \frac{\pi}{4}\log 2\right) - (n-1)\sum_{k=1}^{\infty} \frac{(-1)^k}{k} I(k,n)\right)$$

where $I(k,n) = \Re\left(\int_0^{\frac{\pi}{4}} \theta^{n-1} e^{i2k\theta} \, d\theta\right)$

Proposed by Akerele Olofin-Nigeria

U.1781 Prove that:

$$\int_0^{\infty} \left(\frac{z}{1+2mz+z^2}\right)^{\frac{1}{2}+n} \frac{z+1}{z(z^n+1)} \frac{dz}{\sqrt{z}} = \frac{\Gamma^2(n)}{\Gamma(2n)} \frac{2^{n-1}}{(1+m)^n}$$

Proposed by Hikmat Mammadov-Azerbaijan

U.1782 Prove that:

$$\int_0^1 \frac{z^2 \log\left(\frac{1}{z}\right)}{(1-z^4)\left(\pi^2\left(\log\frac{1+z}{1-z} + 2\tan^{-1}\frac{1}{z}\right)^2\right)} dz = \frac{1}{32}\log\left(\frac{\pi^2}{8}\right)$$

Proposed by Hikmat Mammadov-Azerbaijan

U.1783 If $x, y, z > 0$ and $x \neq y \neq z$ then prove that:

$$\frac{x^2(y-z)}{\log y - \log z} + \frac{y^2(z-x)}{\log z - \log x} + \frac{z^2(x-y)}{\log x - \log y} < \frac{(x+y+z)^2}{8}$$

Proposed by Hikmat Mammadov-Azerbaijan

U.1784 Find:

$$\Omega = \int_0^\infty \left(\frac{1}{\Gamma(1+z)}\left(\frac{z}{e}\right)^z - \frac{1}{\sqrt{2\pi z}}\right)\frac{dz}{e^{(m-\log(1+m))z}}$$

Proposed by Hikmat Mammadov-Azerbaijan

U.1785 Prove that:

$$\int_0^\infty \frac{(5x^2+2x+2)\tan^{-1} x}{(1+x)^2(1+x^2)}dx = \frac{12G + \pi^2 + \pi(5 - \log 8)}{8}$$

where G is Catalan's constant.

Proposed by Ankush Kumar Parcha-India

U.1786 $a = (nx)^2 + (1-mx)^2$ and $b = (nx)^2 + (1+mx)^2$ find:

$$\Omega = \lim_{n\to\infty} \int_0^\infty \tan^{-1}\left(\frac{2nx}{(n^2+m^2)x^2 - 1}\right)\log\left(\frac{a}{b}\right)dx$$

Proposed by Hikmat Mammadov-Azerbaijan

U.1788 If $a, b, c > 0$ and $abc = 1$ then: $a^2 + b^2 + c^2 - 3 \geq 2|(a-1)(b-1)(c-1)|$

Proposed by Hikmat Mammadov-Azerbaijan

U.1789 If $x, y, z \in \mathbb{R}$ then prove that:

$$(x-y)(x-z)(x^2-yz)^2 + (y-z)(y-x)(y^2-zx)^2 + (z-x)(z-y)(z^2-xy)^2 \geq 0$$

Proposed by Hikmat Mammadov-Azerbaijan

U.1790 When x, y and z are numbers forming three sides of a triangle then prove that:

$$\frac{1}{x(x+y-z)} + \frac{1}{y(y+z-x)} + \frac{1}{z(z+x-y)} \geq 3\sqrt{\frac{x+y+z}{xyz(xy+yz+zx)}}$$

Proposed by Hikmat Mammadov-Azerbaijan

U.1791 Find: $\Omega = \int_0^1 \log^2 x \log^2(1-x)\, dx$

Proposed by Togrul Ehmedov-Azerbaijan

U.1792 Find:

$$\Omega=\int_0^1\int_0^1 Li_3(1-xy)\,dxdy$$

Proposed by Togrul Ehmedov-Azerbaijan

U.1793 Find:

$$\Omega=\int_0^1\frac{\log(1-x)\cdot\log(1+x)}{x(1-x^2)}dx$$

Proposed by Togrul Ehmedov-Azerbaijan

U.1794

$$\Omega=\int_0^{\pi}\cos\left(nx-m\tan^{-1}\left(\frac{z\sin x}{1+z\cos x}\right)\right)(1+2z\cos x+z^2)^{-\frac{m}{2}}\left(2\cos\frac{x}{2}\right)^{2n}dx$$

Compare it: $(i)\ \Omega>\pi\ \ (ii)\ \Omega<\pi\ \ (iii)\ \Omega=\pi$.

Proposed by Hikmat Mammadov-Azerbaijan

U.1795

$$\Omega=\int_0^1(\sin^{-1}x)^2\log\frac{1}{1-x^2}\frac{dx}{x^2}+4\int_0^1\sin^{-1}x\log\frac{1}{x}\frac{dx}{1-x^2}$$

Prove that: $\Omega=\frac{\pi^2\log 2}{2}+\frac{\pi^3}{4}-7\zeta(3)$.

Proposed by Hikmat Mammadov-Azerbaijan

U.1796 Prove that: $\int_0^1\frac{dx}{\sqrt{1-x^2}\cdot\sqrt{1-\frac{2-\sqrt{3}}{4}x^2}}dx=\frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right)}{4\sqrt[4]{3}\cdot\sqrt{\pi}}$

Proposed by Hikmat Mammadov-Azerbaijan

U.1797 Prove that:

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\int_0^1\int_0^1\frac{dxdy}{(x^2+y^2)^n}=\frac{2}{3}$$

Proposed by Asmat Qatea-Afghanistan

U.1798 Find:

$$\Omega=\int_0^1\int_0^1\frac{xy}{x+y}\log\left(1+\frac{x+y}{xy}\right)dxdy$$

Proposed by Asmat Qatea-Afghanistan

U.1799 Prove that:

$$\int_0^1 \frac{q^{r-1}(1-q)^{s-1}}{1-pq^a(1-q)^b}dq = \sum_{n=0}^{\infty} \frac{\Gamma(an+r)\Gamma(bn+s)}{\Gamma((a+b)n+r+s)}p^n$$

Proposed by Hikmat Mammadov-Azerbaijan

U.1800 Prove that:

$$\int_0^{\frac{\pi}{2}} e^{2x}\sqrt{\tan x}\,dx = \frac{\sqrt{\pi}(e^{\pi}-1)}{2}\Re\left\{\frac{e^{\frac{i\pi}{4}}\Gamma\left(\frac{e^{\frac{i\pi}{2}}}{2}\right)}{\Gamma\left(\frac{e^{\frac{i\pi}{4}}}{\sqrt{2}}\right)}\right\}$$

Proposed by Balendran Sujeethan-Sri Lanka

U.1801 Prove that:

$$\int_0^{\infty}\left(\frac{\cos^2(\pi x)\,\Gamma\left(\frac{1}{2}+x\right)}{(1+x^2)\Gamma(1+x)} + \frac{\sin(2\pi x)\,\Gamma(x)}{2(1+x^2)\Gamma\left(\frac{1}{2}+x\right)}dx\right) = \Re\left(\frac{\pi(1+e^{-2\pi})\Gamma\left(\frac{1}{2}-i\right)}{2\Gamma(1-i)}\right)$$

Proposed by Balendran Sujeethan-Sri Lanka

U.1802 If $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \sin(2\pi x), u(0,t) = 0, u(2,t) = 0, t > 0, u(x,0) = \sin(\pi x), x \in (0,2)$ then find the function $u(x,t)$.

Proposed by Tobi Joshua-Nigeria

U.1803 Find:

$$\Omega = \int_0^{\infty} \sinh^{-1}(\operatorname{csch} x)\,dx$$

Proposed by Abdul Mukhtar-Nigeria

U.1804 The straight lines 1 and m in $\mathbb{R}^3$ intersect at the origin O, and their direction are parallel to $\langle 1,0,0\rangle$ and $\langle 1,2,0\rangle$, respectively. The point A has position vector given by $\overrightarrow{OA} = \langle 4,1,1\rangle$. The point L on 1 is such that $\overrightarrow{OL}$ is the projection of $\overrightarrow{OA}$ on 1, and the point M on m is such that $\overrightarrow{OM}$ is the projection of $\overrightarrow{OA}$ on m. Show that:

$$\overrightarrow{OL} \equiv \langle 4,0,0\rangle \text{ and } \overrightarrow{OM} = \langle \tfrac{6}{5}, \tfrac{12}{5}, 0\rangle$$

Proposed by Akerele Olofin-Nigeria

U.1805 Let f –be some multiplicative arithmetical function which satisfies the following for all prime p, $\sum_{n=0}^{\infty} f(p^n)x^n = 1 + (p-1)x$. Then prove

$$\sum_{d|n} f(d) = e^{\Lambda(n)}$$

where $n \in \mathbb{N}$ and $\Lambda(\cdot)$ is Mangoldt lambda function.

Proposed by Amrit Awasthi-India

U.1806 Prove that:

$$\frac{\pi^2}{6}(\pi^2-6) = \sum_{n=2}^{\infty}\left[\sin^{-1}\left(\sum_{k=1}^{\infty}\frac{(3^k-1)\Gamma\left(\frac{n+2}{2n}\right)\Phi(-1,k+1,1)}{\sqrt{\pi}\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{n-1}{n}\right)\left(2^{1+2k-\frac{2}{n}}-2^{k+1-\frac{2}{n}}\right)}\right)\right]^2$$

where $\Phi(z,s,a)$ is a Lerch function.

Proposed by Ankush Kumar Parcha-India

U.1807 For $m \in \mathbb{N}$, prove the following:

$$\int_0^1 \frac{\log^{2m} t \log\left(\frac{\sinh t}{t}\right)}{t}dt = \sum_{n=1}^{\infty}\frac{2^{2(n-m-1)}(2m)!}{n^{2(m+1)}(2n)!}B_{2n}$$

where B_n are the Bernoulli numbers.

Proposed by Artan Ajredini-Serbie

U.1808

$$S = \frac{1}{1^3}\cdot\frac{\sinh\left(\frac{\pi}{4}\right)}{\cosh\left(\frac{\pi}{2}\right)} - \frac{1}{3^3}\cdot\frac{\sinh\left(\frac{3\pi}{4}\right)}{\cosh\left(\frac{3\pi}{2}\right)} - \frac{1}{5^3}\cdot\frac{\sinh\left(\frac{5\pi}{4}\right)}{\cosh\left(\frac{5\pi}{2}\right)} + \frac{1}{7^3}\cdot\frac{\sinh\left(\frac{7\pi}{4}\right)}{\cosh\left(\frac{7\pi}{2}\right)} + \cdots$$

Then prove: $S = \frac{\pi^3}{\sqrt{8192}}$.

Proposed by Hikmat Mammadov-Azerbaijan

U.1809 Prove that:

$$\sum_{k>0}\frac{2^{8k}}{3^{3k}k^2\binom{4k}{k}} = \frac{3}{4}\left(\pi^2 - 2\log^2 3 + \left(\tan^{-1}\frac{4\sqrt{2}}{7}\right)^2\right)$$

Proposed by Hikmat Mammadov-Azerbaijan

U.1810

$$\Omega_1 = \int\frac{p(x^2-1)(px^4+2x^3+2x-p)}{[x^2+px(x^2-1)+(x^2-1)^2]^2+p^2(x^2-1)^4}dx$$

$$\Omega_2 = \int\frac{x^2+1}{(px^2+x-p)^2+(x^2-1)^2}dx$$

Evaluate $\Omega = \Omega_1 + \Omega_2$.

Proposed by Ankush Kumar Parcha-India

U.1811 Prove that:

$$\int_0^\infty \log^2 x\,\sqrt{x}e^{-\pi x^2}dx = \frac{\sqrt[4]{\pi}[4\gamma^2 + \log^2(64\pi^2) - 32G + (8\gamma - 4\pi)\log(8\pi) + 5\pi^2]}{16\sqrt{2}\Gamma\left(\frac{1}{4}\right)}$$

where γ —Euler-Mascheroni constant, G —Catalan’s constant.

Proposed by Ankush Kumar Parcha-India

U.1812 $\Omega = e^{\frac{\pi\sqrt{-1}}{6}}\int_0^\infty \frac{dx}{e^{e^{\frac{\pi\sqrt{-1}}{6}}} + e^{-e^{\frac{\pi\sqrt{-1}}{6}}} + e^{e^{\frac{\pi\sqrt{-1}}{6}}\sqrt{-3}}} +$
$\frac{1}{e^{\frac{\pi\sqrt{-1}}{6}}}\int_0^\infty \frac{dx}{\left(e^{\frac{x}{e^{\frac{\pi\sqrt{-1}}{6}}}} e^{-\frac{x}{e^{\frac{\pi\sqrt{-1}}{6}}}} + e^{\frac{\sqrt{-3}x}{e^{\frac{\pi\sqrt{-1}}{6}}}}\right)^2}$

Prove that: $\Omega = \frac{1}{3}$.

Proposed by Hikmat Mammadov-Azerbaijan

U.1813 If $x, y, z > 0$ then prove that:

$$\frac{x}{\sqrt{4y^2 + 4z^2 + yz}} + \frac{y}{\sqrt{4z^2 + 4x^2 + zx}} + \frac{z}{\sqrt{4x^2 + 4y^2 + xy}} \geq 1$$

Proposed by Hikmat Mammadov-Azerbaijan

U.1814 If $x, y, z > 0$ and $x^2 + y^2 + z^2 + xyz = 2(xy + yz + zx)$ then prove that:

$$xy + yz + zx \leq 3(x + y + z) \leq 27$$

Proposed by Hikmat Mammadov-Azerbaijan

U.1815 If $z \geq 0$ and $f''(z) \geq 0$ then:

$$\int_0^z \big(f(2t) - f(t)\big)dt \leq \frac{z\big(f(2z) - f(z)\big)}{2}$$

Proposed by Hikmat Mammadov-Azerbaijan

U.1816

$\Omega_1 = \sum_{n=-\infty}^{\infty} \frac{\cos \pi\sqrt{1+n^2}}{m^2+n^2}, \Omega_2 = \int_{-\infty}^{\infty} \frac{\cos \pi\sqrt{1+z^2}}{m^2+z^2}dz$. Prove that: $\Omega_1 - \Omega_2 = \frac{2\pi \cos \pi\sqrt{1-m^2}}{m(e^{2\pi m}-1)}$

Proposed by Hikmat Mammadov-Azerbaijan

U.1817 Find:

$$\Omega=\int_0^{\infty}\frac{x\cdot(\tan^{-1}x)^2}{(x+1)(x^2+1)}dx$$

Proposed by Togrul Ehmedov-Azerbaijan

U.1818 For $\Re(n)>0$, prove that:

$$\int_0^1\frac{\log x}{x^n+x^{n-1}+\cdots+1}dx=\frac{1}{n^2}\left[\psi^{(1)}\left(\frac{2}{n}\right)-\psi^{(1)}\left(\frac{1}{n}\right)\right]$$

Proposed by Muhamad Afzal-Pakistan

U.1819

$$\Omega=\sum_{m=1}^{\infty}\frac{1}{m^2}\left(\frac{1}{\sinh^2(\pi ma)}+\frac{1}{\frac{\sinh^2(\pi m)}{a}}\right)-8\pi a\sum_{m=1}^{\infty}n^2\log(1-e^{-2\pi ma})-\frac{8\pi}{a}\sum_{m=1}^{\infty}m^2\log\left(1-e^{-\frac{2\pi m}{a}}\right)$$

Prove that: $\Omega=\frac{\pi^2}{30}\left(a^2+\frac{1}{a^2}\right)-\frac{\pi^2}{18}$.

Proposed by Hikmat Mammadov-Azerbaijan

U.1820 If $\left(\sum_{n=0}^{\infty}\frac{H_n}{(n!)^3}x^n\right)\left(\sum_{n=0}^{n}\frac{(-x)^n}{(n!)^3}\right)$ then prove that:

$$\Omega=\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n(3n)!}{(n!)^3\big((2n)!\big)^3}(2H_{2n}+H_n-H_{3n})x^{2n}-\frac{1}{3}\sum_{n=1}^{\infty}\frac{(-1)^n(n!)^3(6n)!}{(6n-1)\big((2n)!\big)^3\big((2n-1)!\big)^3(3n)!}x^2$$

Proposed by Hikmat Mammadov-Azerbaijan

U.1821 Find the maximum real number k that satisfies the following for any $x,y,z>0$:

$$x^6+y^6+z^6-3(xyz)^2\geq k(yz-x)^2(zx-y)^2(xy-z)^2$$

Proposed by Hikmat Mammadov-Azerbaijan

U.1822

$$\Omega=\left(\sum_{n=0}^{\infty}\frac{H_n}{(n!)^2}x_n\right)\left(\sum_{n=0}^{\infty}\frac{(-x)^n}{(n!)^2}\right)$$

Prove that:

$$\Omega=\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n(H_n+H_{2n})}{(n!)^2(2n)!}x^{2n}-\sum_{n=1}^{\infty}\frac{(-1)^n\cdot 2^{4n-3}\cdot n!\cdot(n-1)!}{(2n)!\cdot\big((2n-1)!\big)^2}x^{2n-1}$$

Proposed by Hikmat Mammadov-Azerbaijan

U.1823 Prove that:

$$\sum_{m=0}^{\infty}(-1)^{n-m}\binom{2n}{n-m}m^{2n}=\frac{(2n)!}{2}$$

Proposed by Hikmat Mammadov-Azerbaijan

U.1824 If we have the integral for $a>0$: $\frac{\pi}{a}+\int_0^{\pi}\frac{(1+\sin^2(2x))\cos(4x)}{1+a\sin^2 x}dx=0$

then evaluate the expression: $\Omega=\sqrt[6]{a^5-a^4-a^3-80a^2-144a}$.

Proposed by Srinivasa Raghava-AIRMC-India

U.1825 Let $a,b,c>0, abc=1$. Prove that:

$$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a}\geq\frac{3(a^2+b^2+c^2)}{ab+bc+ca+a^3b^2+b^3c^2+c^3a^2}$$

Proposed by Phan Ngoc Chau-Vietnam

U.1826 Let $a,b,c>0, a^2+b^2+c^2=3abc$. Prove that:

$$\sqrt[3]{\frac{4(b+c)}{a^2}}+\sqrt[3]{\frac{4(c+a)}{b^2}}+\sqrt[3]{\frac{4(a+b)}{c^2}}+3\frac{ab+bc+ca}{a+b+c}\geq 9$$

Proposed by Phan Ngoc Chau-Vietnam

U.1827 Let $a,b,c\geq 0, a^2+b^2+c^2=3$. Prove that:

$$\frac{(a+b)(ab-1)}{ab-3}+\frac{(b+c)(bc-1)}{bc-3}+\frac{(c+a)(ca-1)}{ca-3}\geq 0$$

Proposed by Phan Ngoc Chau-Vietnam

U.1828 Let $a,b,c\in\left(0,\frac{3}{2}\right)$ such that $ab+bc+ca+4=2(a+b+c)+abc$. Find the minimum value of the following expression:

$$P=\frac{a+b+c}{2}+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

Proposed by Phan Ngoc Chau-Vietnam

U.1829 Let $a,b,c>0, a+b+c=2$. Find Max value of the following expression:

$$P=\sqrt{\frac{a^3+1}{4b+1}}+\sqrt{\frac{b^3+1}{4c+1}}+\sqrt{\frac{c^3+1}{4a+1}}$$

Proposed by Phan Ngoc Chau-Vietnam

U.1830 Solve for real numbers: $\sqrt[n]{\frac{11+x}{11-x}}+\sqrt[n]{\frac{11-x}{11+x}}=\sqrt[n]{\frac{2021+x}{2021-x}}+\sqrt[n]{\frac{2021-x}{2021+x}}; n\in\mathbb{N}$

Proposed by Phan Ngoc Chau-Vietnam

U.1831 Prove that:

$$\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n+1}\left(H_{\frac{2n+1}{4}}-H_{\frac{2n-1}{4}}\right)H_n=\frac{7\pi^3}{24}+\frac{5\pi}{2}\log^2 2-4G\log 2-8\Im\big(Li_3(1+i)\big)$$

where $H_n=\int_0^1\frac{1-x^n}{1-x}dx$, G –Catalan's constant, $\sum_{k=0}^{n}\frac{(-1)^{k+1}}{(2k+1)^2}$, $Li_3(z)=\sum_{k=1}^{\infty}\frac{z^k}{k^3}$ is trilogarithm function, $\Im(z)$ is imaginary part of z and i is imaginary part.

Proposed by Naren Bhandari-Nepal

U.1832 Prove that:

$$\sum_{n=0}^{\infty}\frac{\overline{H}_{2n+1}-\frac{1}{n+1}}{16^n(n+1)}\left(\frac{2n+1}{n+1}\binom{2n}{n}\right)^2=\frac{7\pi^2}{3}+20\log^2 2-\frac{32G}{\pi}\log 2-\frac{64}{\pi}\Im\big(Li_3(1+i)\big)$$

-where $\overline{H}_n$ is n^{th} skew-harmonic number, $\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}$, G –Catalan's constant, $\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^2}$, $\Im(z)$ is imaginary part of z, $Li_3(z)$ is trilogarithmic function, $\sum_{k=1}^{\infty}\frac{z^k}{k^3}$ and i is imaginary unit.

Proposed by Naren Bhandari-Nepal

U.1833 Prove that:

$$\lim_{k\to\infty}\sum_{q=0}^{k}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(m+n+2)^{q+3}}=\frac{\pi^2}{6}-1$$

Proposed by Syed Shahabudeen-India

U.1834 $A\in M-4\ (\mathbb{R}), \det A\neq 0, \det A^*=-1, A^*$ –adjoint of A. Prove that:

$$\det(A^2+I_4)\geq(Tr\ A^*)^2$$

Proposed by Marian Ursărescu-Romania

U.1835 Prove that:

$$\cos\frac{\pi}{19}\cos\frac{7\pi}{19}\cos\frac{8\pi}{19}=-\frac{\sqrt{19}}{12}\sin\left[\frac{1}{3}\tan^{-1}\left(\frac{7\sqrt{3}}{9}\right)\right]+\frac{5}{24}$$

Proposed by Vasile Mircea Popa-Romania

U.1836 If $x, y, z \in \mathbb{R}$ such that $x(y+z)>0, y(z+x)>0, z(x+y)>0$,

$xyz(x+y+z)>0$ then:

$$\sqrt{\frac{xy(y+z)(x+z)+yz(x+z)(x+y)+zx(x+y)(y+z)}{xyz(x+y+z)}} \geq \sqrt{\frac{4yz+x(y+z)}{4xz+y(x+z)}}+\sqrt{\frac{4xz+y(x+z)}{4yz+x(y+z)}}$$

Proposed by Bogdan Fuștei-Romania

U.1837 If $a, b, c>0, a+b+c=3$ then: $\frac{a+b+c^2}{a+\sqrt{b}+c}+\frac{a+b^2+c}{\sqrt{a}+b+c}+\frac{a^2+b+c}{a+b+\sqrt{c}} \geq 3$

Proposed by Choy Fai Lam-Hong Kong

U.1838 If $a, b, c>0, a^2+b^2+c^2=3$ then:

$$\frac{a+b+\sqrt{c}}{a+b^2+c}+\frac{a+\sqrt{b}+c}{a^2+b+c}+\frac{\sqrt{a}+b+c}{a+b+c^2} \geq 3$$

Proposed by Choy Fai Lam-Hong Kong

U.1839 $A \in M_4(\mathbb{R}), \det A \neq 0, \det A^*=1, A^*$ –adjoint of A. Prove that:

$$\det(A^2+I_4) \geq (Tr\ A-Tr\ A^*)^2$$

Proposed by Marian Ursărescu-Romania

U.1840 In ΔABC the following relationship holds:

$$(m_a+m_b+m_c)\sqrt{r_a r_b r_c} \geq (r_a+r_b+r_c)\sqrt{h_a h_b h_c}$$

Proposed by Bogdan Fuștei-Romania

U.1841 In acute $\Delta ABC, P$ –point in plane, the following relationship holds:

$$AP \cdot \cos\frac{A}{2}+BP \cdot \cos\frac{B}{2}+CP \cdot \cos\frac{C}{2} \geq 2R+r$$

Proposed by Bogdan Fuștei-Romania

U.1842 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\sqrt{r_a g_a}}{w_a} \geq \left(1+\prod_{cyc}\frac{m_a}{r_a}+\sqrt{\prod_{cyc}\frac{m_a}{h_a}}\right)\sqrt{\frac{2r}{R}}$$

Proposed by Bogdan Fuștei-Romania

U.1843 In ΔABC the following relationship holds:

$$m_a(bc-a^2)+m_b(ac-b^2)+m_c(ab-c^2) \geq 0$$

Proposed by Bogdan Fuștei-Romania

U.1844 In $\Delta ABC, r_1, r_2, r_3$ —Malfatti's radies, the following relationship holds:

$$s+2\sum_{cyc}\sqrt{r_1r_2}\le 4R+r$$

Proposed by Bogdan Fuștei-Romania

U.1845 $P\in Int(\Delta ABC)$ the following relationship holds:

$$\sum_{cyc}\frac{PA}{a}\ge\sqrt{3+\frac{3}{4}\max\left\{\left(\frac{PA}{a}-\frac{PB}{b}\right)^2,\left(\frac{PB}{b}-\frac{PC}{c}\right)^2,\left(\frac{PC}{c}-\frac{PA}{a}\right)^2\right\}}$$

Proposed by Bogdan Fuștei-Romania

U.1846 In ΔABC the following relationship holds:

$$2\max\left\{\frac{m_a}{h_a},\frac{m_b}{h_b},\frac{m_c}{h_c}\right\}\ge\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-1$$

Proposed by Bogdan Fuștei-Romania

U.1847 In ΔABC the following relationship holds:

$$2\max\left\{\frac{m_a}{h_a},\frac{m_b}{h_b},\frac{m_c}{h_c}\right\}\ge\frac{a}{c}+\frac{c}{b}+\frac{b}{a}-1$$

Proposed by Bogdan Fuștei-Romania

U.1848 In $\Delta ABC, \omega$ —Brocard's angle, the following relationship holds:

$$\frac{R}{\sin\omega}>\frac{m_aw_a}{h_a}-\frac{r_a-r}{2}$$

Proposed by Bogdan Fuștei-Romania

U.1849 If $0<a\le b<\frac{\pi}{2}$ then:

$$2\pi\int_a^b\frac{x}{\sin x}dx\le(b-a)\big(2\pi+(\pi-2)(b+a)\big)$$

Proposed by Daniel Sitaru-Romania

U.1850 If $a,b,c\in\mathbb{R}$ with $a\ne 0$ and $a^4-ba^3+2a^2-ca+1$, then prove that

$$b^2+c^2\ge 8.$$

Proposed by Neculai Stanciu-Romania

U.1851 Prove that:

$$\frac{1}{2^n\sqrt{x_1x_2\ldots x_n}}+\sum_{k=1}^{n}\frac{x_k}{(x_1+1)(x_2+1)\cdot\ldots\cdot(x_k+1)}\geq 1,\forall x_i>0,i\in\overline{1,n}$$

Proposed by Neculai Stanciu-Romania

U.1852 In ΔABC the following relationship holds:

$$\frac{1}{(m+n)R^2}\leq\sum_{cyc}\frac{1}{ma^2+nbc}\leq\frac{m+n}{16mnr^2};m,n>0$$

Proposed by Alex Szoros-Romania

U.1853 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\sqrt{m_bm_c}}{h_a}\geq 1+\frac{n_a+n_b+n_c-r}{2R}$$

Proposed by Bogdan Fuștei-Romania

U.1854 If we have the equations: $\begin{cases}\frac{a+b}{2}+\sqrt{ab}=\frac{1+\sqrt{5}}{2}\\ a^2+b^2=\sqrt{5}\end{cases}$

then find the value of $\frac{1}{a}+\frac{1}{b}$.

Proposed by Srinivasa Raghava-AIRMC-India

U.1855 If $x,y,z>0$ then prove:

$$\sum_{cyc}\sqrt{x^4+y^4}+2\sum_{cyc}xy\geq\sum_{cyc}(x^2+y^2)+\sqrt{2}\sum_{cyc}xy$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

U.1856 If $n\in\mathbb{N}^*-\{1\},a\in\mathbb{R},b,c,d,m,p\in\mathbb{R}_+^*,x_k\in\mathbb{R}_+^*,k=\overline{1,n},X_{n,m}=\sum_{k=1}^{n}x_k^m$,

$X_{n,p}=\sum_{k=1}^{n}x_k^p$ such that $c\cdot X_{n,p}>d\cdot\max\limits_{1\leq k\leq n}x_k^p$, then prove that:

$$\sum_{k=1}^{n}\frac{a\cdot X_{n,m}+b\cdot x_k^m}{c\cdot X_{n,p}-d\cdot x_k^p}\geq\frac{n(an+b)}{cn-d}\cdot\frac{X_{n,m}}{X_{n,p}}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

U.1857 If $A_1A_2\ldots A_n,(n\geq 3)$ is a convex polygon with $a_k,k=\overline{1,n}$ the side-lengths and s the semiperimeter, $m\in[1,\infty),S(m)=\sum_{k=1}^{n}a_k^m$, then prove that:

$$\sum_{k=1}^{n} \frac{S_n(m) - a_k^m}{(s-a_k)^m} \geq \frac{2^m n(n-1)}{(n-2)^m}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

U.1858 Prove that:

$$\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{n}{\sinh(nk^2)} = \frac{\pi^2(\pi^4-30)}{360}$$

Proposed by Amrit Awasthi-India

U.1859 In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{\sqrt{m_b m_c}}{r_a} \geq 3$$

Proposed by Bogdan Fuștei-Romania

U.1860

$$Cl_2(\theta) = -\int_0^{\theta} \log\left|2\,sin\left(\frac{t}{2}\right)\right| dt, \theta \in (0,2\pi)$$

Prove: $\Omega = \int_0^{\frac{\pi}{2}} Cl_2(\theta)\, dx = -\frac{35}{32}\zeta(3),$ where $Cl_2(\theta)$ is Clausen function.

Proposed by Togrul Ehmedov-Azerbaijan

U.1861 If $a, b, c > 0$ such that $\frac{3}{a+1} + \frac{3}{b+1} + \frac{3}{c+1} = 8$ then:

$$\prod_{cyc} \left(\sqrt[3]{b} + \sqrt[3]{c}\right) \leq 1$$

Proposed by Marin Chirciu-Romania

U.1862 Let $a, b, c \geq 0$ such that $a + b + c = 3$ then prove:

$$a^{\frac{3}{2}}b + b^{\frac{3}{2}}c + c^{\frac{3}{2}}b + a^{\frac{2021}{2019}}(b+c) + b^{\frac{2021}{2019}}(a+c) + c^{\frac{2021}{2019}}(a+b) \leq 9$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

U.1863 Find a closed form:

$$\Omega = \int_0^{\infty} \frac{1}{(x^2+x+1)(1+ax)} dx, a > 0$$

Proposed by Vasile Mircea Popa-Romania

U.1864 Let $a,b,c>0; ab+bc+ca=a+b+c$ then prove:

$$\frac{\sqrt[3]{4(ab+ac)}}{a^2}+\frac{\sqrt[3]{4(bc+ba)}}{b^2}+\frac{\sqrt[3]{4(ca+cb)}}{c^2}\geq 3\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-1\right)$$

Proposed by Phan Ngoc Chau-Vietnam

U.1865 Let $a,b,c>0$ then:

$$\frac{\frac{1}{a}+b}{\sqrt{\frac{1}{a}+a}}+\frac{\frac{1}{b}+c}{\sqrt{\frac{1}{b}+b}}+\frac{\frac{1}{c}+a}{\sqrt{\frac{1}{c}+c}}\geq 3\sqrt{2}$$

Proposed by Phan Ngoc Chau-Vietnam

U.1866 Let $a,b,c\geq 0; a+b+c=a^2+b^2+c^2$ then prove:

$$2\sqrt{2}\left(\sqrt{ab+ac}+\sqrt{bc+ba}+\sqrt{ca+cb}\right)+9\geq 7(a^2+b^2+c^2)$$

Proposed by Phan Ngoc Chau-Vietnam

U.1867

$$\text{If } \Omega_1=\int_0^\infty\frac{x(x-1)}{(x^3-1)[a^2x^2+(1+x)^2]}dx,\Omega_2=\int_0^\infty\frac{x^2(x^2+x-2)}{(x^3-1)[(a^2x^2+x+1)^2+x^4]}dx$$

then:

$$\Omega_1+\Omega_2=\frac{\pi}{2a}-\frac{2\pi}{3\sqrt{3}}+(a^2-1)\left[\frac{2\pi a^2+3\sqrt{3}\log a-\pi}{3\sqrt{3}(a^4-a^2+1)}-\frac{\pi(a^2-1)}{2a(a^4-a^2+1)}\right],$$

$$\forall\, a\in\mathbb{R}-\{0\}$$

Proposed by Ankush Kumar Parcha-India

U.1868 Find:

$$\Omega=\int_0^1 x^3\tan^{-1}\left(1+\frac{\sqrt{2}}{x}\right)dx$$

Proposed by Asmat Qatea-Afghanistan

U.1869 Let $a,b,c>0, ab+bc+ca=3$ then prove:

$$\frac{1}{abc}-1\geq\frac{1}{24}\left(\frac{(a-b)^2}{a^2-ab+b^2}+\frac{(b-c)^2}{b^2-bc+c^2}+\frac{(c-a)^2}{c^2-ca+a^2}\right)$$

Proposed by Phan Ngoc Chau-Vietnam

U.1870 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{1}{a}(\sin B+\sin C)\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)=\frac{1}{2RF}\sum_{cyc}a^2$$

Proposed by Ertan Yildirim- Turkiye

U.1871 In $\Delta ABC, x, y, z > 0$ the following relationship holds:

$$\sum_{cyc}\frac{n_a^2}{h_a^2}\cdot x+\frac{xr_a+yr_b+zr_c}{r}\ge x+y+z+\frac{s}{r}\sqrt{xy+yz+zx}$$

Proposed by Bogdan Fuștei-Romania

U.1872 In $\Delta ABC, G_e$ –Gergonne's point, the following relationship holds:

$$\sum_{cyc}r_a\cdot\sum_{cyc}AG_e\ge\sum_{cyc}\frac{m_aw_a(r_b+r_c)}{n_a}$$

Proposed by Bogdan Fuștei-Romania

U.1873 If $a, b, c > 0$ then:

$$\frac{1}{(a+1)^2}+\frac{1}{(b+1)^2}+\frac{1}{(c+1)^2}+\frac{ab+bc+ca}{8}\ge\frac{9}{8}$$

Proposed by Hoang Le Nhat Tung-Vietnam

U.1874 If $x, y, z > 0, x+y+z+2=xyz$ then:

$$\frac{1}{3x+y+4}+\frac{1}{3y+z+4}+\frac{1}{3z+x+4}\le\frac{1}{4}$$

Proposed by Hoang Le Nhat Tung-Vietnam

U.1875 Let $a, b, c > 0$ such that

$\frac{1}{2a^2+bc}+\frac{1}{2b^2+ca}+\frac{1}{2c^2+ab}\ge 1$ then prove:

$$a+b+c\ge ab+bc+ca$$

Proposed by Hung Nguyen Viet-Vietnam

U.1876 Let $a, b, c > 0$ such that $a+b+c > 0$ then prove:

$$\sqrt{3(a^2+b^2+c^2)}+\frac{3(ab+bc+ca)}{a+b+c}\le 2(a+b+c)$$

Proposed by Hung Nguyen Viet-Vietnam

U.1877 Let $a, b, c > 0$ such that $a + b + c > 0$ then prove:

$$\sqrt{a^2+b^2+c^2}+\left(3-\sqrt{3}\right)\frac{ab+bc+ca}{a+b+c}\geq a+b+c$$

Proposed by Hung Nguyen Viet-Vietnam

U.1878 Find:

$$\Omega=\lim_{x\to 0}\left(\frac{1}{x^2}\sum_{k=1}^{n}\left(\frac{x}{k}\right)^{2k}\right)^{\frac{1}{x^2}}$$

Proposed by Hussain Reza Zadah-Afghanistan

U.1879 Find without any software: $\Omega=\sin\left(\frac{5\pi}{13}\right)\sin\left(\frac{10\pi}{13}\right)\sin\left(\frac{15\pi}{13}\right)\sin\left(\frac{20\pi}{13}\right)\sin\left(\frac{25\pi}{13}\right)$

Proposed by Hussain Reza Zadah- Afghanistan

U.1880 Find:

$$\Omega=\lim_{n\to\infty}\frac{\sqrt[n-1]{(n-3)!!\,(n-1)(n-2)!!}}{n-1}$$

Proposed by Jay Jay Oweifa-Nigeria

U.1881 Find: $\Omega=\lim_{n\to\infty}\sqrt[2n]{\frac{n^n n!!(n-1)!!}{(2n-1)!!(2^n n!)}}\cdot\frac{n}{\sqrt[n]{(2n)!!}}$

Proposed by Jay Jay Oweifa-Nigeria

U.1882 In ΔABC the following relationship holds:

$$(a^3+b^3+c^3)\left(\frac{a}{4s^2-a^2}+\frac{b}{4s^2-b^2}+\frac{c}{4s^2-c^2}\right)\geq\frac{27\sqrt{3}}{32}F$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

U.1883 Find a closed form:

$$\Omega=\int_0^\infty\frac{dx}{(z^2+x^2)(x^{4\pi x}-1)};\Re(z)>1$$

Proposed by Kaushik Mahanta-India

U.1884 Prove that:

$$\int_0^1\frac{x^2\log(x^2)}{x^4+x^2+1}dx=-\frac{1}{6}\zeta(2)-\frac{1}{36}\psi^{(1)}\left(\frac{1}{3}\right)+\frac{1}{9}\psi^{(1)}\left(\frac{2}{3}\right)+\frac{1}{36}\psi^{(1)}\left(\frac{5}{6}\right)$$

Proposed by Max Wong-Hong Kong

U.1885 In $\Delta ABC, G$ –centroid, $GD \parallel AC, GE \parallel BA, GF \parallel CB, D \in (BC), E \in (CA), F \in (AB)$. If ρ –inradii of ΔDEF then:

$$\rho = \frac{[ABC]}{m_a + m_b + m_c}$$

Proposed by Mehmet Șahin-Turkiye

U.1886 V –Bevan's point in $\Delta ABC, I_a, I_b, I_c$ –excenters, $VK \perp (I_bI_c), K \in (I_bI_c), VL \perp (I_cI_a), L \in (I_cI_a), VM \perp (I_aI_b), M \in (I_aI_b)$. Prove that:

$$\frac{a}{bc \cdot VK^2} + \frac{b}{ca \cdot VL^2} + \frac{c}{ab \cdot VM^2} = \frac{r_a + r_b + r_c}{2R^2F}$$

Proposed by Mehmet Șahin-Turkiye

U.1887 In $\Delta ABC, AD, BE, CF$ –Nagel's cevians. Prove that:

$$2(DE + EF + FD) \le \sqrt{3(41R^2 - 125r^2)}$$

Proposed by Mehmet Șahin-Turkiye

U.1888 Solve for real numbers:

$$\frac{x-8}{x+4+4\sqrt{x+1}} \le \frac{3\sqrt{16-x}}{x}$$

Proposed by Miguel Velasquez Culque-Peru

U.1889 Solve that system:

$$\begin{cases} x^3 + \dfrac{3x-11}{2} + |x-3|\sqrt{x+3} = y^2 + 8y + 8 \\ |x-3| = \sqrt{2y - 5x + 11} \end{cases}$$

Proposed by Minh Nhat Nguyen-Vietnam

U.1890 In ΔABC the following relationship holds:

$$\left(\sum_{cyc} \frac{m_a}{h_a}\right)^n \left(\sum_{cyc} \frac{h_a}{m_a}\right)^m \le \frac{3^{2m}}{2^{2m-n}} \left(\frac{R}{r}\right)^{2m} \left(1 + \left(\frac{R}{r}\right)^n\right); m, n \in \mathbb{N}^*$$

Proposed by Mokhtar Khassani-Algerie

U.1891 Prove that:

$$\lim_{n\to\infty} 2^n n^{2n} e^{-n} \left(\left(1 + \frac{1}{n+1}\right)^{n+1} - \left(1 + \frac{1}{n}\right)^n\right)^n = \frac{1}{\sqrt[6]{e^{17}}}$$

Proposed by Mokhtar Khassani-Algerie

U.1892 Find all values $\mu \in \mathbb{R}$ such in ΔABC the following relationship holds:

$$\frac{|a-b|}{m_a h_a w_a}+\frac{|b-c|}{m_b h_b w_b}+\frac{|c-a|}{m_c h_c w_c} \le \mu(a-b(b-c)(c-a)|$$

Proposed by Mokhtar Khassani-Algerie

U.1893 If $a,b>0$ then:

$$\frac{2a+b}{a+b}>\frac{2}{3}\left(1+\frac{b}{a}\right)^{\frac{a}{b}}$$

Proposed by Mohammed Bouras-Morocco

U.1894 Solve for real numbers t:

$$\int_0^1 \frac{1}{x}\log(1+x)\left[\log(1+x^2)-t\cdot\log(1-x)\right]dx=\frac{\pi}{2}\cdot G$$

G –Catalan's constant.

Proposed by Mohammed Bouras-Morocco

U.1895 Solve the system:

$$\begin{cases} x^2+xy+yz=m \\ y^2+yz+xz=m \\ z^2+xz+xy=m \end{cases}$$

Proposed by Carlos Paiva-Brazil

U.1896 Solve for real numbers: $x^5-4x^3+3x-1=0$

Proposed by Carlos Paiva-Brazil

U.1897 O –is in ΔABC, $\{Y\}=AO\cap BC$: $\overrightarrow{AD}=\frac{1}{2}\overrightarrow{AY}$, $\{Z\}=BO\cap BC$ and E: $\overrightarrow{BE}=\frac{1}{2}\overrightarrow{BZ}$, $\{X\}=CO\cap BC$ and F: $\overrightarrow{CF}=\frac{1}{2}\overrightarrow{CX}$, M is in ΔXYZ, $\{A'\}=MA\cap ZX$, $\{B'\}=MB\cap XY$, $\{C'\}=MC\cap YZ$. Prove: $[A'B'C']\le[DEF]$.

Proposed by Dang Le Gia Khanh-Vietnam

U.1898 If $a_k\ge 0, k\in\overline{1,n}, 0<x<\frac{\pi}{2}$ then:

$$\sum_{k=1}^{n}\frac{a_k^4}{\sin^{3k}x}\ge\left(\frac{1-\sin x}{\sin x}\right)^3\left(\sum_{k=1}^{n}a_k\right)^4$$

Proposed by Emil Popa, Mihaela Duță-Romania

U.1899 In acute ΔABC, O –circumcentre, I –incentre, I_a, I_b, I_c –excenters, $\Delta A'B'C'$ –orthic triangle. Prove that:

$$8r < \sum_{cyc} \frac{B'C'}{OI_a} \cdot \sum_{cyc} II_a < \frac{9\sqrt{3}R^2}{2r}$$

Proposed by Radu Diaconu-Romania

U.1900 In ΔABC the following relationship holds:

$$8 \leq \frac{1}{abc} \prod_{cyc} \left(\frac{h_a}{\sin A} + \frac{h_b}{\sin B} \right) \leq \left(\frac{R}{r} \right)^3$$

Proposed by Ertan Yildirim-Turkiye

U.1901 In ΔABC the following relationship holds: $\frac{h_a - r}{h_a + r_a} + \frac{h_b - r}{h_b + r_b} + \frac{h_c - r}{h_c + r_c} \leq \frac{R}{2r}$

Proposed by Ertan Yildirim-Turkiye

U.1902

$$\Omega = \cos\left(\sqrt{\frac{\pi^2}{4} - 2\int_0^1 \frac{log(1+mx)}{x\sqrt{1-x^2}} dx} \right), m > 0$$

$$A. \Omega < m, \quad B. \Omega = m, \quad C. \Omega > m$$

Proposed by Ghazaly Abiodun-Nigeria

U.1903

$$\text{If } \int_0^1 \frac{\log(1+x)}{x} [\log(1+x^2) - t\log(1-x)] dx = \frac{\pi}{2} C \text{ then prove:}$$

$$t = \frac{8}{5A} \sum_{n=1}^{\infty} \frac{(-1)^n (H_n - H_{2n})}{n^2} - \frac{4\pi C}{5A}$$

where A is Apery's constant and C —Catalan's constant.

Proposed by Abdul Hafeez Ayinde-Nigeria

U.1904 In $\Delta ABC, P, Q$ in plane of ΔABC such that $\beta\overrightarrow{AB} + \gamma\overrightarrow{BP} + \overrightarrow{PC} = 0$ and $\overrightarrow{AQ} + \alpha\overrightarrow{QB} + \overrightarrow{BC} = 0, \alpha, \beta, \gamma \in \mathbb{R}, \alpha \neq 1, \gamma \neq 1$. Prove that A, P and Q are collinear if and only if $\alpha + \gamma = \beta + 1$.

Proposed by Florică Anastase-Romania

U.1905 Solve for natural numbers:

$$\begin{cases} (x+y+z)^2 + xy + yz + zx = 16(x+y+z) + 56 \\ \big(x(y+z) + y(z+x) + z(x+y)\big)^2 = (x+1)(10z+1)^2 \end{cases}$$

Proposed by Mokhtar Khassani-Algerie

U.1906 In ΔAC the following relationship holds:

$$\frac{\cos A}{h_a}+\frac{\cos B}{h_b}+\frac{\cos C}{h_c}\le\left(\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\right)\left(\frac{r_a}{h_a}+\frac{r_b}{h_b}+\frac{r_c}{h_c}\right)$$

Proposed by Mokhtar Khassani-Algerie

U.1907 Solve for natural numbers: $\begin{cases}x^3+y^3=8(xy+z^2+y)+z\\ x^2+y^2+z^2=xyz-10x\\ x+y-z=8\end{cases}$

Proposed by Mokhtar Khassani-Algerie

U.1908 Find a closed form:

$$\Omega=\prod_{n=1}^{\infty}\left(1+\frac{(-1)^{n+1}}{3n-1}\right)$$

Proposed by Abdul Mukhtar-Nigeria

U.1909 Solve for real numbers: $\begin{cases}2xy-3y+1=0\\ \left(x+\frac{x^3}{y^2}\right)\left(x-\frac{x^3}{y^2}+\frac{1}{y}\right)=2\end{cases}$

Proposed by Orlando Irahola Ortega-Peru

U.1910 In ΔABC the following relationship holds:

$$\max\{r_a,r_b,r_c\}\ge\min\{h_a,h_b,h_c\},\qquad \min\{r_a,r_b,r_c\}\le\max\{h_a,h_b,h_c\}$$

Proposed by Rahim Shahbazov-Azerbaijan

U.1911

$$x_1=1,x_{n+1}=\frac{x_n}{2x_n+3},n\ge 1$$

Find:

$$\Omega=\lim_{n\to\infty}\frac{1}{3^n\cdot x_n}$$

Proposed by Rajeev Rastogi-India

U.1912 $0<x,y,z<1,(1-x)(1-y)(1-z)=xyz$. Find:

$$\Omega=\min\left(\frac{1-x}{xy},\frac{1-y}{yz},\frac{1-z}{zx}\right)$$

Proposed by Rajeev Rastogi-India

U.1913 In ΔABC the following relationship holds:

$$\frac{a^2b^2+b62\,c^2+c^2a^2}{abc(a+b+c)} \leq \frac{R}{2r}+\frac{s^2+2Rr+r^2-8R^2}{2s^2}$$

Proposed by Soumava Chakraborty- India

U.1914 Let $\omega = e^{\frac{i\pi}{4}} = \frac{1+i}{\sqrt{2}}$. Show that:

$$\frac{1}{1+i}\operatorname{erf}\left(\omega x\sqrt{\frac{\pi}{2}}\right) = \int_0^x e^{-it^2\frac{\pi}{2}}\,dt = C(x) - iS(x)$$

Proposed by Nawar Alasadi-Iraq

U.1915 In acute ΔABC the following relationship holds:

$$\frac{1}{\tan A}+\frac{1}{\tan B}+\frac{1}{\tan C} \geq \frac{\cos B\cos C}{\sin A}+\frac{\cos C\cos A}{\sin B}+\frac{\cos A\cos B}{\sin C}$$

Proposed by Marin Chirciu-Romania

U.1916 Let $a,b,c>0$ and $\lambda \geq \frac{9}{8}$. Prove that:

$$\sum_{cyc}\sqrt{\frac{2ab}{a^2+b^2}}+\frac{3\lambda(a^2+b^2+c^2)}{ab+bc+ca} \leq 3(\lambda+1)$$

Proposed by Marin Chirciu-Romania

U.1917 In ΔABC the following relationship holds:

$$\frac{3r}{R} \leq \sum_{cyc}\frac{r_a}{h_b+h_c} \leq \frac{3R}{4r}$$

Proposed by Marin Chirciu-Romania

U.1918 In ΔABC the following relationship holds: $\left(\frac{3r}{2R}\right)^2 \leq \sum_{cyc}\frac{\sin B\sin C}{\csc^2\frac{A}{2}} \leq \frac{9r}{8R}$

Proposed by Marin Chirciu-Romania

U.1919 In ΔABC the following relationship holds:

$$\sum_{cyc}a^2\left(\frac{1}{b^2}+\frac{1}{c^2}\right)+6(n-1) \geq n\sum_{cyc}\left(\frac{b}{c}+\frac{c}{b}\right), n \leq \frac{3}{4}$$

Proposed by Marin Chirciu-Romania

U.1920 If $x,y,z>0, x\sqrt{yz}+y\sqrt{zx}+z\sqrt{xy}$ and $n \in \mathbb{N}^*$, then find the minimum of

$$P = x^{2n}+y^{2n}+z^{2n}$$

Proposed by Marin Chirciu-Romania

U.1921 Let $\lambda, k, n \in \mathbb{R}$, fixed. Solve for real numbers:

$$(\lambda x - 1)\sqrt{x^4 + k^2 + 2kn^2} = \lambda x^3 - x^2 + \lambda kx - k$$

Proposed by Marin Chirciu-Romania

U.1922 In ΔABC the following relationship holds:

$$\frac{1}{27}\left(\frac{7R-2r}{2R}\right)^4 \le \frac{m_a^4}{h_a^4} + \frac{m_b^4}{h_b^4} + \frac{m_c^4}{h_c^4} \le \frac{3}{64}\left(\frac{27R^2}{4r^2} - 19\right)^2$$

Proposed by Marin Chirciu-Romania

U.1923 In ΔABC the following relationship holds:

$$4\left(\sum_{cyc}\frac{1}{bc(1-\cos A)}\right)^2 = \left(\sum_{cyc}\frac{1}{h_a}\right)^2\left(\sum_{cyc}\frac{1}{r_a}\right)^2$$

Proposed by Marin Chirciu-Romania

U.1924 In ΔABC the following relationship holds:

$$\mu\sum_{cyc} a(b^2 + c^2) \ge a^3 + b^3 + c^3 + (6\mu - 3)abc, \qquad \mu \ge 2$$

Proposed by Marin Chirciu-Romania

U.1925 In ΔABC the following relationship holds: $\sum_{cyc}\frac{1}{r_a h_a^2} \ge \frac{2}{9Rr^2}$

Proposed by Marin Chirciu-Romania

U.1926 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\cos^2\frac{B-C}{2}}{\cot\frac{A}{2}} \ge \frac{2\sqrt{3}r}{R}$$

Proposed by Marin Chirciu-Romania

U.1927 In ΔABC the following relationship holds:

$$\frac{\sin B\sin C}{h_a} + \frac{\sin C\sin A}{h_b} + \frac{\sin A\sin B}{h_c} \le \frac{1}{4}\sum_{cyc} m_a \sum_{cyc}\frac{r_a}{h_a}$$

Proposed by Marin Chirciu-Romania

U.1928 Find:

$$\Omega = \int_{\frac{1}{a}}^{a} \frac{x^{n(k+1)-1} \cdot \log x}{(1 + x^{2k})^{n+1}} dx, a > 0, n \in \mathbb{N}, k \in \mathbb{N} - \{0\}$$

Proposed by Marin Chirciu-Romania

U.1929 In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{b + c}{2s + a} \geq \frac{3}{2} + \frac{\mu r(R - 2r)}{Rs}, \mu \leq \frac{5}{64}$$

Proposed by Marin Chirciu-Romania

U.1930 Solve for real numbers:

$$\sqrt{4\mu x - y^2} = \sqrt{4\mu x^2 + y} + \sqrt{y + \mu + 1}, \mu > 0 - \text{fixed.}$$

Proposed by Marin Chirciu-Romania

U.1931 In ΔABC the following relationship holds:

$$\mu + \frac{\sin A \sin B \sin C}{\sin A + \sin B + \sin C} \geq \frac{4\mu + 1}{9} \sum_{cyc} \sin^2 A, \mu \geq 2$$

Proposed by Marin Chirciu-Romania

U.1932 In ΔABC the following relationship holds:

$$\mu + \frac{\cot A \cot B \cot C}{\cot A + \cot B + \cot C} \leq \frac{9\mu + 1}{9} \sum_{cyc} \cot^2 A, \mu \in \mathbb{R}$$

Proposed by Marin Chirciu-Romania

U.1933 Let $x, y \geq 0, x + y = 2, 1 \leq a \leq 4, a -$fixed. Find: $\min \Omega(x, y), \max \Omega(x, y)$.

$$\Omega = \frac{1}{x^2 - ax + 2a - 1} + \frac{1}{y^2 - ay + 2a - 1}$$

Proposed by Marin Chirciu-Romania

U.1934 In acute ΔABC the following relationship holds:

$$\sum_{cyc} \frac{1}{\tan A (\mu + \tan A)} \geq \frac{9}{3 + 2\mu\sqrt{3}}, \mu \geq 0$$

Proposed by Marin Chirciu-Romania

U.1935 In ΔABC the following relationship holds:

$$\sum_{cyc}\sqrt{\frac{2h_bh_c}{h_b^2+h_c^2}}\geq\frac{3h_ah_bh_c}{m_am_bm_c}$$

Proposed by Marin Chirciu-Romania

U.1936 In ΔABC the following relationship holds:

$$\sqrt[3]{\frac{h_a}{r_a}}+\sqrt[3]{\frac{h_b}{r_b}}+\sqrt[3]{\frac{h_c}{r_c}}\geq\frac{6r}{R}$$

Proposed by Marin Chirciu-Romania

U.1937 Let $a,b,c>0, a+b+c=3$. Prove that:

$$3+2\left(\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}\right)\geq\sum_{cyc}\sqrt{1+4\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}$$

Proposed by Phan Ngoc Chau-Vietnam

U.1938 Find:

$$\Omega=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\sec y\sqrt{\frac{1+\sqrt{1+\frac{\tan^4 y}{4}\tan^2 y\sin^2 x}}{1+\frac{\tan^4 y}{4}\tan^2 x\sin^2 x}}\,dxdy$$

Proposed by Balendran Sujeethan-Sri Lanka

U.1939 Find:

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\left[\sum_{i=1}^{n}\left(i+\frac{1}{4}\right)\right]^{-1}$$

Proposed by Vasile Mircea Popa-Romania

U.1940 Find a closed form:

$$\Omega=\sum_{n=1}^{\infty}\frac{n}{(2n+1)\cdot 4^n}$$

Proposed by Vincenzo Dima-Italy

U.1941 Solve for real numbers:

$$\frac{x}{y}+\frac{5}{x}+\frac{y-5}{5}=\frac{y+x}{y+5}+\frac{5+y}{5+x}$$

Proposed by Daniel Sitaru-Romania

U.1942

$$\Omega(x)=\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x+1}{2}\right)\Gamma\left(\frac{1-x}{2}\right)\Gamma\left(\frac{2-x}{2}\right)\sin(\pi x), \qquad 0<x<1$$

Solve for real numbers:

$$x^2-\frac{4x}{\Omega(x)}+\frac{1}{\pi^4}=0$$

Proposed by Daniel Sitaru-Romania

U.1943 If $a,b,c,x,y>0$ then:

$$\sum_{cyc(a,b,c)}\left(\frac{\sqrt{a^{x+y}}}{\sqrt{b^{x+y}}+\sqrt{c^{x+y}}}-\frac{a^{\sqrt{xy}}}{b^{\sqrt{xy}}+c^{\sqrt{xy}}}\right)\ge 0$$

Proposed by Daniel Sitaru-Romania

U.1944 If $a,b,c>0$ then:

$$8\left(\sum_{cyc}\frac{a^5}{b^2+c^2}\right)\left(\sum_{cyc}\frac{a^6}{b^3+c^3}\right)\left(\sum_{cyc}\frac{a^7}{b^4+c^4}\right)\ge (a^3+b^3+c^3)^3$$

Proposed by Daniel Sitaru-Romania

U.1945

$$\Omega(\alpha,\beta)=\int_{-1}^{1}\frac{(1+x)^{2\alpha-1}(1-x)^{2\beta-1}}{(1+x^2)^{\alpha+\beta}}dx\,,\alpha,\beta>0$$

Find a closed form and prove that:

$$\Omega(3,5)>\sqrt{\Omega(4,5)\cdot\Omega(3,6)}$$

Proposed by Daniel Sitaru-Romania

U.1946 Find a closed form:

$$\Omega=\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\int_{\sin^{-1}\varepsilon}^{\sin^{-1}(1-\varepsilon)}\log\left((\cos x)^{\cot x}\cdot(\sin x)^{\frac{\cos x}{1+\sin x}}\right)dx$$

Proposed by Daniel Sitaru-Romania

U.1947 Find all values $x, y, z > 0$ such that:

$$\begin{cases} x + y + 2z = 6 \\ \frac{3}{y}\left(\frac{2}{x} + \frac{1}{y}\right) = 4\left(\frac{2}{x+y} + \frac{1}{2y}\right)^2 \\ x + 2^y + \log_2 z = 4 \end{cases}$$

Proposed by Daniel Sitaru-Romania

U.1948 If $a, b, c > 0$ then:

$$\frac{\frac{a(b^3+c^3)}{b+c} + \frac{b(c^3+a^3)}{c+a} + \frac{c(a^3+b^3)}{a+b}}{\frac{a(b^2+c^2)}{b+c} + \frac{b(c^2+a^2)}{c+a} + \frac{c(a^2+b^2)}{a+b}} \le \frac{a^3 + b^3 + c^3}{a^2 + b^2 + c^2}$$

Proposed by Daniel Sitaru-Romania

U.1949 a, b, c –sides in ΔABC, $\sqrt{a}, \sqrt{b}, \sqrt{c}$ –sides in $\Delta A'B'C'$. Prove that:

$$\frac{1}{r_a r'_a} + \frac{1}{r_b r'_b} + \frac{1}{r_c r'_c} = \frac{aa' + bb' + cc'}{FF'}$$

Proposed by Mehmet Șahin-Turkiye

U.1950 Let $a, b, c \ge 0, abc = 1$. Prove that:

$$\sqrt{\frac{a}{a + 6b + 2bc}} + \sqrt{\frac{b}{c + 6c + 2ca}} + \sqrt{\frac{c}{c + 6a + 2ab}} \ge 1$$

Proposed by Phan Ngoc Chau-Vietnam

U.1951 Let $a, b, c \ge 0, a + b + c = 3$. Prove that:

$$3 \le \sqrt{a^2 - ab + b^2} + \sqrt{b^2 - bc + c^2} + \sqrt{c^2 - ca + a^2} \le 6$$

Proposed by Phan Ngoc Chau-Vietnam

U.1952 In ΔABC the following relationship holds:

$$\sqrt[3]{\prod_{cyc}\left(\sin^4\frac{A}{2} + \sin^4\frac{B}{2}\right)} \ge \frac{1}{6}\left(\frac{2r}{R} - \left(\frac{r}{R}\right)^2\right)$$

Proposed by Marian Ursărescu-Romania

U.1953 For $a_1, a_2, \dots, a_n \ge 0, n \in \mathbb{N}^*$ and k is positive real number. Prove:

$$(a_1^2 + k)(a_1^2 + a_2^2 + k) \cdot \ldots \cdot (a_1^2 + a_2^2 + \cdots + a_n^2 + k) \ge \sqrt{k^n(n+1)^{n+1}} a_1 a_2 \cdot \ldots \cdot a_n$$

Proposed by Phan Ngoc Chau-Vietnam

U.1954 Let $a,b,c\geq \frac{2}{3}, a+b+c=9$. Prove that: $\sqrt{ab+bc+ca}\leq \sqrt{a}+\sqrt{b}+\sqrt{c}$

Proposed by Phan Ngoc Chau-Vietnam

U.1955 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{b+c-a}{2a\sqrt{bc}}\geq \frac{25r^2-4R^2}{3\sqrt{3}R^3}$$

Proposed by Mehmet Șahin-Turkiye

U.1956 Prove that: $\frac{1}{3}\sum_{k=1}^{\infty}\left[\sum_{n=0}^{k}\frac{1}{n!}\right]k2^{-k}=\sqrt{e}$

Proposed by Vincenzo Dima-Italy

U.1957 Prove that:

$$\int_0^1\int_0^1\int_0^1\sqrt{x^2+y^2+z^2}\,dxdydz=\frac{\sqrt{3}}{4}+\log\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)-\frac{\pi}{24}$$

Proposed by Asmat Qatea-Afghanistan

U.1958 Prove that:

$$16\sum_{k=1}^{\infty}\left[(-1)^{k-1}\left(\sum_{j=1}^{k}(2j-1)^{-1}\right)\right]k^{-1}=\pi^2$$

Proposed by Vincenzo Dima-Italy

U.1959 If $\Omega=\int_0^1 x^2\log\left(\frac{1+\sqrt{1-x^2}}{x}\right)\sin^{-1}(bx)\,dx$ then show:

$$\Omega=\frac{1}{12}\big(Li_2(b)-Li_2(-b)\big)+\frac{1}{72b^3}\left(b(b^2-9)\log\left(\frac{1+b}{1-b}\right)-8\log(1-b^2)+10b^2\right)$$

Proposed by Ose Favour-Nigeria

U.1960 Prove that: $\int_0^1\frac{Li_3(-x)}{1+x^2}dx=-\frac{3\pi}{128}\zeta(3)-\beta(4)+\frac{1}{2}\zeta(2)\beta(2)$

Proposed by Daniel Immarube-Nigeria

U.1361 If $x_i>0, i=\overline{1,n}$ such that $x_1+x_2+\cdots+x_n=1$ and f is convex, then:

$$(i)\ \sum_{i=1}^{n}f(x_i)\geq\sum_{i=1}^{n}f\left(\frac{1-x_i}{n-1}\right)\qquad (ii)\ \frac{(1-x_1)(1-x_2)\cdot\ldots\cdot(1-x_n)}{x_1x_2\cdot\ldots\cdot x_n}\geq(n-1)^n$$

Proposed by Neculai Stanciu-Romania

U.1962 In ΔABC the following relationship holds:

$$\sqrt[3]{\prod_{cyc}\left(\cos^4\frac{A}{2}+\cos^4\frac{B}{2}\right)}\ge\frac{1}{6}\left(4+\frac{5r}{R}-\left(\frac{r}{R}\right)^2\right)$$

Proposed by Marian Ursărescu-Romania

U.1963 In ΔABC the following relationship holds:

$$\sum_{cyc}a^2\ge 4\sqrt{3}F+\frac{11rR-2r}{12}$$

Proposed by Nguyen Van Canh-Vietnam

U.1964 Let $a,b,c\ge 0$. Prove that:

$$(a+b+c)\left(\frac{2a+1}{a+1}+\frac{2b+1}{b+1}+\frac{2c+1}{c+1}\right)+9\ge 3\sum_{cyc}\sqrt[3]{ab+bc+1}$$

Proposed by Phan Ngoc Chau-Vietnam

U.1965 If in $\Delta ABC, I$ $-$incenter, I_a,I_b,I_c $-$excenters, then:

$$\frac{I_aI_b}{IC}+\frac{I_bI_c}{IA}+\frac{I_cI_a}{IB}=\frac{a+b+c}{r}$$

Proposed by Ertan Yildirim-Turkiye

U.1966 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{(r_a+r_b)(r_a+r_c)}{h_b+h_c}\ge\frac{9r}{2}$$

Proposed by Ertan Yildirim-Turkiye

U.1967 In ΔABC the following relationship holds: $\sum_{cyc}\frac{h_b^2+h_c^2}{m_a^2+m_b^2-c^2}=\frac{1}{R^2}\sum_{cyc}a^2$

Proposed by Ertan Yildirim-Turkiye

U.1968 For $x,y,z>0, xyz=1$ and $x^3>36$ prove: $\frac{x^2}{3}+y^2+z^2>xy+yz+zx$

Proposed by Hikmat Mammadov-Azerbaijan

U.1969 Show that the equation $k^2=4mn-m-n$ has no solution in natural numbers.

Proposed by Hikmat Mammadov-Azerbaijan

U.1970 Find:

$$\Omega=\int\frac{(x^2-1)\sqrt{x^2\left(x^2+2x-1+\frac{2}{x}+\frac{1}{x^2}\right)}}{x^2(x+1)^2}dx$$

Proposed by Hikmat Mammadov-Azerbaijan

U.1971 Find:

$$\Omega=\sum_{n=0}^{\infty}\left(\sum_{k=1}^{\infty}(-1)^{k-1}\int_0^{\infty}e^{-(n+k)x}\,dx\right)^2$$

Proposed by Hikmat Mammadov-Azerbaijan

U.1972

$$\text{If}\int\frac{dx}{\sin^3 x-\cos^3 x}=M\tan^{-1}(h(x))+N\log\left|\frac{\sqrt{2}+h(x)}{\sqrt{2}-h(x)}\right|+C$$

where $h(x)=\sin x+\cos x$, then find: $\Omega=12M+9\sqrt{2}N-3$.

Proposed by Hikmat Mammadov-Azerbaijan

U.1973 Show that:

$$\int_0^{\frac{\pi}{4}}\frac{x\sin x}{1+\sqrt{2}\cos x}dx=\frac{1}{8\sqrt{2}}(8G-3\pi\log 2)$$

Proposed by Ose Favour-Nigeria

U.1974 Prove that: $\int_0^{\frac{\pi}{2}}\left(\frac{1}{\log(tan\,x)}+\frac{2-\sqrt[3]{\tan^4 x}}{1-\tan x}\right)dx=\pi$

Proposed by Hikmat Mammadov-Azerbaijan

U.1975 Let ΔABC and $\Delta A_1B_1C_1$ −with sides a,b,c and a_1b_1,c_1, respectively. F,F_1 −areas, $P\in Int(ABC),R_a,R_b,R_c$ −circumradii of $\Delta BPC,\Delta APC,\Delta APB$. Prove that:

$$\frac{a_1}{AP}+\frac{b_1}{BP}+\frac{c_1}{CP}\geq\sqrt{\frac{1}{2}\sum_{cyc}\left(\frac{a}{BP\cdot CP}\right)^2(b_1^2+c_1^2-a_1^2)+\frac{FF_1}{2RR_aR_bR_c}}$$

Proposed by Bogdan Fuștei-Romania

U.1976 Prove that:

$$\int_{-\infty}^{\infty}\log(2-2\,cos^2\,x)\,dx=-\sqrt{2\pi}\zeta\left(\frac{3}{2}\right)$$

Proposed by Hikmat Mammadov-Azerbaijan

U.1977 Prove that:

$$\sum_{k=0}^{\infty}\frac{\sin\left(\frac{k\pi}{4}\right)}{k!\sqrt{2^k}}\pi^k=\sqrt{e^{\pi}}$$

Proposed by Vincenzo Dima-Italy

U.1978 In acute $\Delta ABC, AD, BE, CF$ −altitudes, r_1, r_2, r_3 −inradii of $\Delta ABD, \Delta BCE, \Delta CAF$. Prove that: $F=\frac{1}{3}(ar_1+br_2+cr_3+rr_a+rr_b+rr_c)$

Proposed by Mehmet Șahin-Turkiye

U.1979 Let R_a, R_b, R_c be circumradii of $\Delta BGC, \Delta CGA, \Delta AGB, G$ −centroid in ΔABC. Prove that:

$$\frac{1}{R_a^2}+\frac{1}{R_b^2}+\frac{1}{R_c^2}\geq\frac{16F^2}{9R^6}$$

Proposed by Mehmet Șahin-Turkiye

U.1980 Find:

$$\Omega=\lim_{n\to\infty}H_n\left(\sum_{k=1}^{n}\frac{1}{n+k}-\sum_{k=1}^{n}\cot^{-1}(n+k)\right)$$

Proposed by Mokhtar Khassani-Algerie

U.1981 In ΔABC the following relationship holds:

$$\frac{1}{w_a}\int_{h_a}^{w_a}\frac{\sin x}{x}dx+\frac{1}{w_b}\int_{h_b}^{w_b}\frac{\sin x}{x}dx+\frac{1}{w_c}\int_{h_c}^{w_c}\frac{\sin x}{x}dx\leq\frac{R-2r}{r}$$

Proposed by Mokhtar Khassani-Algerie

U.1982 Solve for real numbers:

$$64\sin^2 x\sin^4(2x)\,64\cos^2(2x)\cos^4 x=5(3+\cot x)$$

Proposed by Mokhtar Khassani-Algerie

U.1983 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{w_b+w_c}{w_a\left(\sqrt{h_b}+\sqrt{h_c}\right)}\geq\frac{3\sqrt{2R}(\sum w_a)^2}{(\sum r_a)\left(\sum b\cos\frac{A}{2}\right)}$$

Proposed by Mokhtar Khassani-Algerie

U.1984 If $a, b, c\geq 1$ and $\lambda\geq 0$ then:

$$\frac{a^2bc}{\sqrt{bc}+\lambda}+\frac{b^2ca}{\sqrt{ca}+\lambda}+\frac{c^2ab}{\sqrt{ab}+\lambda}\geq\frac{3}{\lambda+1}$$

Proposed by Marin Chirciu-Romania

U.1985 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{m_a}{s_a}+\prod_{cyc}\frac{w_a}{h_a}\geq 4$$

Proposed by Marin Chirciu-Romania

U.1986 If $x,y,z>0, x+y+z=1$ and $-9\leq\lambda\leq 3$ then:

$$\frac{(\sum yz)^2}{xyz}+\lambda\sum_{cyc}yz\geq 3+\frac{\lambda}{3}$$

Proposed by Marin Chirciu-Romania

U.1987 If $x,y,z>0, xy+yz+zx=1$ and $-9\leq\lambda\leq 3$ then:

$$(x+y+z)^2+\lambda xyz(x+y+z)\geq 3+\frac{\lambda}{3}$$

Proposed by Marin Chirciu-Romania

U.1988 In ΔABC the following relationship holds:

$$\sum_{cyc}\sqrt[3]{\frac{\csc^4\frac{A}{2}}{\csc^2\frac{B}{2}+\csc\frac{C}{2}\left(\csc\frac{A}{2}+\csc\frac{B}{2}\right)}}\geq\sqrt[3]{36}$$

Proposed by Marin Chirciu-Romania

U.1989 Let $a>1, b>1$ fixed. Solve for real numbers:

$$a^x+b^{\frac{2}{x}}+a^x\cdot b^{\frac{2}{x}}=a^2(1+b)+b$$

Proposed by Marin Chirciu-Romania

U.1990 Let $a,b>0$ fixed. Solve for real numbers:

$$2\sqrt{\frac{2x}{ax^2+(a+b+2)x+b}}-\sqrt{\frac{x}{ax^2+(a+b+1)x+b}}=1$$

Proposed by Marin Chirciu-Romania

U.1991 If $a,b,c>0, \sum a=\sum a^2$ and $\lambda\geq 0$, then:

$$\frac{a}{a+\lambda b}+\frac{b}{b+\lambda c}+\frac{c}{c+\lambda a}\ge\frac{a+b+c}{\lambda+1}$$

Proposed by Marin Chirciu-Romania

U.1992 In ΔABC the following relationship holds:

$$\frac{\sin\frac{3A}{2}}{\cos\frac{A}{2}}+\frac{\sin\frac{3B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{3C}{2}}{\cos\frac{C}{2}}\le\frac{9R}{s}$$

Proposed by Marin Chirciu-Romania

U.1993 Be p a prime number, arbitrary. Solve on positive integers (x,y,z):

$$\begin{cases}xy+z^2=3p+4\\x+yz^2=p+4\end{cases}$$

Proposed by Marin Chirciu-Romania

U.1994 If $x,y,z>0$ then:

$$\sum_{cyc}\frac{(x+y)^4}{x^4+14x^2y^2+y^4}\ge 3$$

Proposed by Marin Chirciu-Romania

U.1995 If $x,y,z>0,\frac{1}{1+x}+\frac{1}{1+y}+\frac{1}{1+z}=1$ then: $x+y+z\ge\frac{3}{4}xyz$

Proposed by Marin Chirciu-Romania

U.1996 In ΔABC the following relationship holds:

$$3R\le\sum_{cyc}\frac{h_a}{\sin^2 A}\le\frac{3R^4}{8r^3}$$

Proposed by Marin Chirciu-Romania

U.1997 If $a_1,a_2,\dots,a_n>0,\frac{1}{a_1+\lambda}+\frac{1}{a_2+\lambda}+\cdots+\frac{1}{a_n+\lambda}=\frac{n}{\lambda+1}$ and $\lambda\ge 0$ then:

$$a_1+a_2+\cdots+a_n\ge n$$

Proposed by Marin Chirciu-Romania

U.1998 In ΔABC the following relationship holds:

$$\frac{a^{2n+1}}{m_a}+\frac{b^{2m+1}}{m_b}+\frac{c^{2n+1}}{m_c}\ge 2\sqrt{3}(6Rr)^n, n\in\mathbb{N}$$

Proposed by Marin Chirciu-Romania

U.1999 In acute ΔABC the following relationship holds:

$$\sum_{cyc}\left(\frac{b+c}{\cos A}\right)^n \geq 3\left(\frac{8s}{3}\right)^n, n \in \mathbb{N}$$

Proposed by Marin Chirciu-Romania

U.2000 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{h_a^2}{r_b r_c} \leq \sum_{cyc}\frac{r_a^2}{h_b h_c}$$

Proposed by Marin Chirciu-Romania

U.2001 In ΔABC the following relationship holds:

$$\frac{m_a^2 m_b^2 + m_b^2 m_c^2 + m_c^2 m_a^2}{m_a^2 + m_b^2 + m_c^2} \leq \frac{9R^2}{4}$$

Proposed by Marin Chirciu-Romania

U.2002 Solve for real numbers: $\begin{cases} x^2 = yz + 1 \\ y^2 = zx + a \\ z^2 = xy + a^2 \end{cases}, a > 1$

Proposed by Marin Chirciu-Romania

U.2003 In acute ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\tan^5 A}{\tan^3 B} \geq \frac{1}{2}\left(\frac{s}{r}\right)^2$$

Proposed by Marin Chirciu-Romania

U.2004 If $a, b, c > 0, abc = 1, 0 \leq n \leq 2$ then:

$$a^3 + b^3 + c^3 + n\sum_{cyc}\frac{ab}{a^2+b^2} \geq \frac{3}{2}(n+2)$$

Proposed by Marin Chirciu-Romania

U.2005 If $x, y, z > 0, \lambda \geq \frac{7}{8}$ then:

$$\lambda\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)+\frac{8xyz}{(x+y)(y+z)(z+x)} \geq 3\lambda + 1$$

Proposed by Marin Chirciu-Romania

U.2006 We have for $n \geq 1$

$$\sum_{m=0}^{n} (-1)^m \ {}_2F_1\left(\frac{1}{2},\frac{1-m}{2};\frac{3}{2};\cos^m\left(\frac{\pi}{3}\right)\right) = 2^n - 2^{\frac{n}{2}}\sqrt{2^n-1}$$

where ${}_2F_1(a,b;c;x)$ is Hypergeometric function.

Proposed by Srinivasa Raghava-AIRMC-India

U.2007 Prove that:

$$\sum_{n=1}^{\infty} \frac{H_{\left[\frac{n}{2}\right]}}{4^n n}\binom{2n}{n} = \sum_{n=1}^{\infty} \frac{H_n}{4^{2n-1}}\left(\frac{2}{2n-1} - \frac{2}{4n-1} + \frac{1}{8n}\right)\left[\binom{4n}{2n}\right]$$

where H_n is nth harmonic number and $[x]$ is Ceiling function.

Proposed by Narendra Bhandari-Nepal

U.2008 Solve the system: $\begin{cases} x^2y - xy^2 - 1 = \frac{y^3}{\varphi} \\ y^2x - yx^2 - 1 = \frac{x^3}{\varphi} \end{cases}$, where $\varphi -$ Golden ratio.

Proposed by Srinivasa Raghava-AIRMC-India

U.2009 Solve the system:

$$\begin{cases} x^2 + x\sin y + \cos y = \dfrac{1}{2} \\ \sin^{-1}\left(\dfrac{x}{2} + \sin y\right) = y + \dfrac{\pi}{3} \end{cases}$$

Proposed by Srinivasa Raghava-India

U.2010 If we have the integral

$$\beta = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sqrt[3]{x} - \frac{1}{\sqrt[3]{x}}}{\sqrt[3]{x} + \frac{1}{\sqrt[3]{x}}} \frac{dx}{1+x+x^2}$$

then prove the relation: $9\beta^6 + 396\beta^4 + 432\beta^2 + 64 = 0$.

Proposed by Srinivasa Raghava-AIRMC-India

U.2011 If we define:

$$S(x,n) = \frac{(-1)^n \cdot n!}{x^{n+1}} \cdot \sum_{k=0}^{n} \frac{(-1)^k}{k!} \cdot \sin\left(\frac{k\pi}{2} + x\right) \cdot x^k$$

then prove that:

$$\lim_{n\to\infty}\sum_{n=0}^{2m}\lim_{x\to 0} S(x,n)=\frac{\pi}{4}$$

Proposed by Amrit Awasthi-India

U.2012 If $n\in\mathbb{N}$, then

$$\sum_{k=2}^{\infty}\frac{\cos\left(\frac{k\pi}{2}\right)(\zeta(k)-\zeta(k,n)-1)}{k}=\log(\Gamma(n))-\frac{1}{2}\log\left(\frac{sinh(\pi)|\Gamma(n+i)|^2}{2\pi}\right)$$

where $\zeta(k,n)$ is the Hurwitz zeta function.

Proposed by Angad Singh-India

U.2013 Find:

$$\Omega=\int_0^\infty\int_0^\infty\int_0^\infty\sum_{n=1}^{\infty}\frac{yz^{\sqrt{\frac{n+1}{n}}}(1+y)\sqrt{n+1}}{(1+z)^{\sqrt{\frac{n+1}{n}}}(1+x)^2(1+y)^2(1+z)^2}\left[\frac{xy}{(1+x)(1+y)}\right]^n dxdydz$$

Proposed by Ankush Kumar Parcha-India

U.2014 Find:

$$\Omega=\int_0^\infty\frac{x\log x\log^3(x+1)}{(x+1)^3}dx$$

Proposed by Artan Ajredini-Serbie

U.2015 Prove that:

$$\int_0^1\int_0^1 x^2y^2\sqrt{x^2+y^2}\,dxdy=\frac{3\sqrt{2}-\log(1+\sqrt{2})}{28}$$

Proposed by Asmat Qatea-Afghanistan

U.2016 Prove that:

$$\sum_{n=1}^{\infty}\frac{1}{256^n n^2}\binom{8n}{4n}=\frac{11\pi^2}{12}-11\log^2 2-4Li_2\left(\frac{1}{\sqrt{2}}\right)+2Li_2\left(\frac{2-\sqrt{2}}{4}\right)+$$

$$+3\log 2\log(1+\sqrt{2})+Li_2\left(\frac{1-\sqrt{2}}{2}\right)-\frac{3}{2}\log^2(1+\sqrt{2})\,4Li_2\left(-\sqrt{\frac{1+\sqrt{2}}{2}}\right)+$$

$$+2Li_2\left(\frac{\sqrt{2}-\sqrt{1+\sqrt{2}}}{2\sqrt{2}}\right)-\log^2\left(\sqrt{2}+\sqrt{1+\sqrt{2}}\right)-3\log 2\log\left(\sqrt{1+\sqrt{2}}-\sqrt{2}\right)$$

where $Li_2(x)$ denotes dilogarithm function.

Proposed by Narendra Bhandari-Nepal

U.2017 Find: $\Omega = \int_0^1 \int_0^1 \int_0^1 \log(1-x)\log(1-xy)\log(1-xyz)\,dxdydz$

Proposed by Togrul Ehmedov-Azerbaijan

U.2018 Find a closed form:

$$\Omega(a,b) = \int_0^1 \frac{\sin(a\,log\,x)\sin(b\log x)}{x\log^2 x}dx, \qquad a,b \in \mathbb{R}$$

Proposed by Togrul Ehmedov-Azerbaijan

U.2019 In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{m_a}{(m_b+m_c)^2} \geq \frac{1}{2R}$$

Proposed by Marian Ursărescu-Romania

U.2020 Prove that for $a > 0$, the following relationship holds:

$$\int_1^\infty \frac{x^2\tan^{-1}(ax)}{x^4+x^2+1}dx \cong \frac{\pi^2}{8\sqrt{3}} + \frac{\pi}{3}\log 3 - \frac{\pi}{6a\sqrt{2}} + \frac{1}{3a^3}\left(\frac{\log 3}{4} - \frac{\pi}{12\sqrt{3}}\right) -$$

$$-\frac{1}{5a^5}\left(\frac{1}{2} - \frac{\pi}{12\sqrt{2}} - \frac{\log 3}{4}\right) + \frac{1}{7a^7}\left(\frac{\pi}{6\sqrt{3}} - \frac{1}{2}\right) - \frac{1}{108a^9} +$$

$$+\frac{1}{9a^9}\left(\frac{\log 3}{9} - \frac{\pi}{12\sqrt{3}}\right) - \frac{1}{11a^{11}}\left(\frac{11}{24} - \frac{\pi}{12\sqrt{3}} - \frac{\log 3}{4}\right) + \cdots$$

Proposed by Narendra Bhandari-Nepal

U.2021 Solve for real numbers:

$$\left[\frac{2\tan x}{1+\tan x}\right] = 2 - \left[\frac{2\cos x}{\sin x + \cos x}\right], [*] - \text{GIF}.$$

Proposed by Hikmat Mammadov-Azerbaijan

U.2022 Let f be a class C^3 function define on $[0,1]$ such that:

$$I_n = \int_0^{\frac{1}{n}} xf'(nx)\,dx + \frac{1}{2n^4}\sum_{k=0}^{n-1}\left(f'\left(\frac{k}{n}\right) + 2nf\left(\frac{k}{n}\right)\right), \qquad I_n = \frac{A}{n^2} + \frac{B}{n^4} + O\left(\frac{1}{n^4}\right)$$

Find A and B.

Proposed by Serlea Kabay-Liberia

U.2023 Find a closed form:

$$\Omega=\int_0^{\infty}\left(\frac{z}{1+2mz+z^2}\right)^{\frac{1}{2}+n}\frac{z+1}{z(z^n+1)}\frac{dz}{\sqrt{z}}$$

Proposed by Hikmat Mammadov-Azerbaijan

U.2024 Prove that for $n\in\mathbb{N}$

$$\int_0^{\infty}xe^{-\frac{x^2}{2}}I_{2n-1}(x)\,dx=\frac{1}{2^{n-\frac{1}{2}}}\left(\frac{{}_1F_1\left(n+\frac{1}{2};2n;\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(2n)}\right)$$

where ${}_1F_1(a;b;z)$ is confluent Hypergeometric function of first kind, $I_v(z)$ is modified function of first kind.

Proposed by Kaushik Mahanta-India

U.2025 Prove the following equality holds:

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{4^n}\binom{2n}{n}\frac{\overline{H_{2n}}}{2n}=$$

$$=2Li_2(-1-\sqrt{2})-Li_2(1+\sqrt{2})+Li_2(2+\sqrt{2})+Li_2(\sqrt{2})-2Li_2(-\sqrt{2})+$$

$$+\frac{\log^2 2}{2}+\frac{3}{2}\log\left(\frac{1}{2}\right)\log\left(\frac{2}{1+\delta_S}\right)-\log(2+\sqrt{2})\log(2+2\sqrt{2})+2\log^2(1+\delta_S)-$$

$$-2\log 2\tanh^{-1}(\delta_S)+Li_2\left(-\frac{1}{\delta_S}\right)-\frac{\pi^2}{6}+\log^2\left(\frac{2}{1+\delta_S}\right)-\log\left(\frac{1}{2}\right)\log\left(\frac{1}{\delta_S}\right)$$

where $\delta_S=1+\sqrt{2}$ is Silver ratio, $Li_2(x)$ is dilogarithm function and $\overline{H_n}$ is nth Skew Harmonic number.

Proposed by Narendra Bhandari-Nepal

U.2026 Find:

$\Omega=\sum_{k=1}^{\infty}\frac{H_{2k}(-1)^{k-1}}{2k+1}$, where H_n is the nth harmonic number.

Proposed by Vincenzo Dima-Italy

U.2027 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{r_b r_c}{w_a^2}=\frac{1}{2}\left(3+\frac{m_a}{s_a}+\frac{m_b}{s_b}+\frac{m_c}{s_c}\right)$$

Proposed by Bogdan Fuștei-Romania

U.2028 In ΔABC the following relationship holds:

$$\frac{g_a}{w_a}+\frac{g_b}{w_b}+\frac{g_c}{w_c}\ge\sqrt{\frac{r}{2R}}\sum_{cyc}\frac{r_a+r_b}{n_c}$$

Proposed by Bogdan Fuștei-Romania

U.2029 In ΔABC the following relationship holds:

$$\frac{1}{2}\sum_{cyc}\tan\frac{A}{4}\ge\sum_{cyc}\sqrt{\frac{m_a}{r_a}}-\frac{a+b+c}{4r}$$

Proposed by Bogdan Fuștei-Romania

U.2030 In acute $\Delta ABC, AD, BE, CF-$ altitudes, $r_1, r_2, r_3, r_4, r_5, r_6-$ inradii of $\Delta ABD, \Delta ACD, \Delta BCE, \Delta BAE, \Delta ACF$ $-$respectively ΔBCF.Prove that:

$$|(r_1-r_2)(r_3-r_4)(r_5-r_6)|=\frac{s}{32R}|(a-b)(b-c)(c-a)|$$

Proposed by Mehmet Șahin-Turkiye

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

ROMANIAN MATHEMATICAL MAGAZINE-R.M.M.-SPRING 2024

PROBLEMS FOR JUNIORS

JP.466 If $a, b, c, d \in \mathbb{R}$ such that $(a^2+b^2)(c^2+d^2)=25$ then:

$$3bd+4ad+4bc \leq 3ac+25$$

Proposed by Daniel Sitaru-Romania

JP.467 If $x, y > 0$ then in ΔABC the following relationship holds:

$$\frac{xa^n+yb^n}{xa^{n-1}+yb^{n-1}}+\frac{xb^n+yc^n}{xb^{n-1}+yc^{n-1}}+\frac{xc^n+ya^n}{xc^{n-1}+yb^{n-1}} \geq a+b+c$$

Proposed by Daniel Sitaru-Romania

JP.468 Solve for real numbers: $x^{12}-15x^3+14=0$

Proposed by Daniel Sitaru-Romania

JP.469 Let $z_1, z_2, z_3 \in \mathbb{C}^*, A(z_1), B(z_2), C(z_3)$ different in pairs such that

$|z_1|=|z_2|=|z_3|=1$. If

$$\sum_{cyc}\sqrt{(2z_1-z_2-z_3)(2z_2-z_1-z_3)}=9 \Rightarrow AB=BC=CA.$$

Proposed by Marian Ursărescu-Romania

JP.470 Let $z_1, z_2, z_3 \in \mathbb{C}, A(z_1), B(z_2), C(z_3)$ different in pairs such that

$|z_1|=|z_2|=|z_3|=1$. If

$$\sum_{cyc}\frac{1}{(|(z_1-z_2)|z_1-z_3|+(z_1-z_3)|z_1-z_2||)^2}=\frac{3}{(|z_1-z_2|+|z_2-z_3|+|z_3-z_1|)^2}$$

$$\Leftrightarrow AB=BC=CA.$$

Proposed by Marian Ursărescu-Romania

JP.471 In $\Delta ABC, AA_1, BB_1, CC_1$ internal bisectors and A_2, B_2, C_2 contact points with circumcircle of triangle ABC. Prove that:

$$A_1A_2 \cdot B_2C_2 + B_1B_2 \cdot A_2C_2 + C_1C_2 \cdot A_2B_2 \geq Rs$$

Proposed by Marian Ursărescu-Romania

JP.472 If $x, y, z > 0$ then:

$$\left(3x^3 - \frac{1}{x^2} + \frac{1}{x^5}\right)\left(3y^3 - \frac{1}{y^2} + \frac{1}{y^5}\right)\left(3z^3 - \frac{1}{z^2} + \frac{1}{z^5}\right) \geq (xy + yz + zx)^3$$

Proposed by Daniel Sitaru-Romania

JP.473 If $a_k > 0, k = \overline{1,5}$ then prove that exists $i, j \in \overline{1,5}$ such that:

$$0 \leq \frac{a_j - a_i}{1 + a_i a_j} \leq \sqrt{2} - 1$$

Proposed by Daniel Sitaru-Romania

JP.474 If $0 < b \leq a$ then:

$$\sqrt{a^2 + ab} + \sqrt{a^2 + \left(\frac{a+b}{2}\right)^2} \leq 2a + \left(\sqrt{2} - 1\right)\left(\sqrt{ab} + \frac{a+b}{2}\right)$$

Proposed by Daniel Sitaru-Romania

JP.475 If $x, y, z > 0$ such that $x + y + z = 3$ and $\lambda \geq 0$ then:

$$\text{(i)}\ \frac{1}{(x+\lambda)^2} + \frac{1}{(y+\lambda)^2} + \frac{1}{(z+\lambda)^2} \geq \frac{3}{(\lambda+1)^2}$$

$$\text{(ii)}\ \frac{x}{(y+\lambda)^2} + \frac{y}{(z+\lambda)^2} + \frac{z}{(x+\lambda)^2} \geq \frac{3}{(\lambda+1)^2}$$

Proposed by Marin Chirciu-Romania

JP.476 In ΔABC the following relationship holds:

$$\frac{a}{(b+\lambda c)^{n+1} s_a^n} + \frac{b}{(c+\lambda a)^{n+1} s_b^n} + \frac{c}{(a+\lambda b)^{n+1} s_c^n} \geq \frac{3}{(\lambda+1)^{n+1}}\left(\frac{1}{sR}\right)^n, \lambda \geq 0, n \in \mathbb{N}$$

Proposed by Marin Chirciu-Romania

JP.477 Let $a > 1, b > 1$ fixed. Solve for real numbers:

$$a^{\log_{2b}\left(x + \frac{b^2}{x}\right)} = \frac{(a+2b)x - b^2 - x^2}{x}$$

Proposed by Marin Chirciu-Romania

JP.478 Let $m, n \geq 0$ and $ABC, A_1B_1C_1$ triangles with areas F, F_1 respectively, then

$$\frac{a^{m+2} \cdot a_1^{n+1}}{h_a^m} + \frac{b^{m+2} \cdot b_1^{n+1}}{h_b^n} + \frac{c^{m+2} \cdot c_1^{n+1}}{h_c^m} \geq 2^{m+n+1} \cdot \left(\sqrt[4]{3}\right)^{1-2m-n} \cdot F \cdot \left(\sqrt{F_1}\right)^{n+1}$$

Proposed by D.M. Bătinețu-Giurgiu, Constantin Cocea-Romania

JP.479 If $a, b, c, d > 0; ab = cd; a < b, c < d; x, y \in [a, b]$ and $y, t \in [c, d]$, then:

$$ab(x + y + z + t)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{t}\right) \leq (a + b + c + d)^2$$

Proposed by Daniel Sitaru-Romania

JP.480 In ΔABC the following relationship holds:

$$\sum_{cyc} \sqrt{\cot^2 \frac{A}{2} + \cot^2 \frac{B}{2} + 3} \geq 9$$

Proposed by Marin Chirciu-Romania

PROBLEMS FOR SENIORS

SP.466 Let $A, B \in M_4(\mathbb{R})$. If $AB + BA = O_4$ then $\det(A^4 + A^2 + B^2) \geq 0$

Proposed by Marian Ursărescu-Romania

SP.467 Let $z_1, z_2, z_3 \in \mathbb{C}^*, A(z_1), B(z_2), C(z_3)$ different in pairs such that

$$|z_1| = |z_2| = |z_3|. \text{ If } \left|\frac{z_1+z_2}{z_1-z_2}\right|^2 + \left|\frac{z_2+z_3}{z_2-z_3}\right|^2 + \left|\frac{z_3+z_1}{z_3-z_1}\right|^2 = 1 \Rightarrow AB = BC = CA.$$

Proposed by Marian Ursărescu-Romania

SP.468 In ΔABC the following relationship holds:

$$\frac{3\sqrt{3}}{2}k \leq \sum_{cyc} \frac{\sin^2 A}{\sin B + \sin C} \leq \frac{\sqrt{6}}{12}\left(\frac{4}{k} + 1\right)\sqrt{1 - k}, \qquad k \in \left(0, \frac{1}{2}\right]$$

Proposed by George Apostolopoulos- Greece

SP.469 In ΔABC the following relationship holds:

$$\frac{y + z}{x \cdot w_a^4} + \frac{z + y}{y \cdot w_b^4} + \frac{x + y}{z \cdot w_c^4} \geq \frac{32}{27R^4}, \qquad x, y, z > 0$$

Proposed by Marin Chirciu-Romania

SP.470 In $\Delta ABC, O_a$ –circumcevian, holds:

$$\frac{6r}{R}\le\frac{r_a}{o_a}+\frac{r_b}{o_b}+\frac{r_c}{o_c}\le\frac{2R}{r}-1$$

Proposed by Marin Chirciu-Romania

SP.471 In ΔABC the following relationship holds:

$$\frac{3}{2}\sqrt[6]{\frac{4r^5}{R^2}}\le\sum_{cyc}\sqrt{m_a\cos\frac{B}{2}\cos\frac{C}{2}}\le\frac{4R+r}{\sqrt{2R}}$$

Proposed by Marin Chirciu-Romania

SP.472 If $x,y,z>0, x+y+z=1$ then:

$$\left(x+\frac{1}{y}\right)^5+\left(y+\frac{1}{z}\right)^5+\left(z+\frac{1}{x}\right)^5\ge\frac{100.000}{81}$$

Proposed by Daniel Sitaru-Romania

SP.473 If $x,y,z>0, \Delta ABC$ and $A_1\in(BC), B_1\in(CA), C_1\in(AB)$ such that

$A_1B=xA_1C, B_1C=yB_1A, C_1A=zC_1B$, then holds:

$$aa_1+bb_1+cc_1\ge4\sqrt{3}\cdot\sqrt{\frac{xyz+1}{(x+1)(y+1)(z+1)}}\cdot F$$

Proposed by D.M. Bătinețu-Giurgiu, Mihaly Bencze-Romania

SP.474 If $m\ge0$ and $x,y,z>0$ then in ΔABC holds:

$$\frac{x^{m+1}\cdot a^{2m}}{(y+z)^{m+1}\cdot h_a^2}+\frac{y^{m+1}\cdot b^{2m}}{(z+x)^{m+1}\cdot h_b^2}+\frac{z^{m+1}\cdot c^{2m}}{(x+y)^{m+1}\cdot h_c^2}\ge2^{m-1}\cdot\left(\sqrt{3}\right)^{1-m}\cdot F^{m-1}$$

Proposed by D.M. Bătinețu-Giurgiu, Mihaly Bencze-Romania

SP.475 In ΔABC the following relationship holds:

$$\frac{m_a^2\cdot a^3}{\sqrt{m_bm_c}}+\frac{m_b^2\cdot b^3}{\sqrt{m_cm_a}}+\frac{m_c^2\cdot c^3}{\sqrt{m_am_b}}\ge8\sqrt{3}\cdot F^2$$

Proposed by D.M. Bătinețu-Giurgiu, Mihaly Bencze-Romania

SP.476 If $x,y,z>0$ and $0\le\lambda\le\frac{1}{25}$ then:

$$\sum_{cyc}\frac{x}{\sqrt[3]{\lambda y^3+xyz+\lambda z^3}}\ge\frac{3}{\sqrt[3]{2\lambda+1}}$$

Proposed by Marin Chirciu-Romania

SP.477 If $a,b,c,d\ge 1$ then: $\sqrt{a-1}+\sqrt{b-1}+\sqrt{c-1}+\sqrt{d-1}\le\sqrt{2(ab+cd)}$

Proposed by Daniel Sitaru-Romania

SP.478 Let a,b,c –be the sides lengths of ΔABC, I –incenter, G –centroid. If $IG\perp BC$ and $b\ne c$ then:

$$\frac{b}{c+a}+\frac{c}{a+b}+\frac{ab+bc+ca}{a^2+b^2+c^2}<\frac{13}{6}$$

Proposed by Florică Anastase-Romania

SP.479 In ΔABC the following relationship holds:

$$\frac{R^2}{r}\ge\frac{s}{16}\left(\frac{a}{w_b}+\frac{b}{w_a}\right)\left(\frac{b}{w_c}++\frac{c}{w_b}\right)\left(\frac{a}{w_a}+\frac{c}{w_c}\right)\ge\frac{8r^2}{R}$$

Proposed by Alex Szoros-Romania

SP.480 In $\Delta ABC, F$ –area and the points $M\in(BC), N\in(CA), P\in(AB)$, then:

$$(a^2+b\cdot BM)(b^2+c\cdot CP)(c^2+a\cdot AM)\ge 36\sqrt{3}\cdot F^3$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

UNDERGRADUATE PROBLEMS

UP.466 Find:

$$\Omega=\int_0^{\pi}\left(\frac{x\cos x}{1+\sin x}\right)^2dx$$

Proposed by Florică Anastase-Romania

UP.467 Find:

$$\Omega=\int_0^{\pi}\frac{x^2\cos^3 x}{(1+\sin^2 x)^2}dx$$

Proposed by Florică Anastase-Romania

UP.468 Let $m \in \mathbb{N}$ and $0 < x < m$. If $i = \left[\frac{(m+1)x}{x+1}\right]$, prove that

$2\binom{m}{i} \geq x^{m-i}$, where $[x]$ is integer part of x and $\binom{m}{i}$ is a binomial coefficient.

Proposed by Ovidiu Pop-Romania

UP.469 Prove that:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{(n!)^2}} < \frac{\pi^2 e}{6}$$

Proposed by Daniel Sitaru-Romania

UP.470 If $(a_n)_{n\geq 1}, a_0 > 0, a_{n+1} = (2n+1)a_n, \forall n \in \mathbb{N}^*$ then find:

$$\Omega = \lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}\right)$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

UP.471 Let $m \geq 0$ and $H_n = \sum_{k=1}^{n} \frac{1}{n}, n \in \mathbb{N}^*$. Find:

$$\Omega = \lim_{n\to\infty}\left(\left(\sqrt[n+1]{(n+1)!}\right)^{m+1} - \left(\sqrt[n]{n!}\right)^{m+1}\right) \cdot e^{-mH_n}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

UP.472 Find:

$$\Omega = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} e^x \cdot \sin x\,(x + x\cot x + 1)\,dx$$

Proposed by D.M. Bătinețu-Giurgiu, Florică Anastase-Romania

UP.473 Find:

$$\Omega = \lim_{n\to\infty}\left(\sqrt[n+1]{\frac{\left((n+1)!\right)^2}{(2n+1)!!}} - \sqrt[n]{\frac{(n!)^2}{(2n-1)!!}}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

UP.474 If $(a_n)_{n\geq 1}, a_n \in \mathbb{R}_+^* = (0,\infty), n \in \mathbb{N}^*$ and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n \cdot \sqrt[n]{n!}} = a > 0$. Find:

$$\Omega(\mathrm{a}) = \lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

UP.475 Let $(a_n)_{n\ge1}$ be sequence of real numbers such that $a_1 = 1$ and $(n+1)^2(a_{n+1}-a_n)-(a_{n+1}+n+1)=0$. Find:

$$\Omega = \lim_{n\to\infty} \sqrt[n]{1+a_n}$$

Proposed by Florică Anastase-Romania

UP.476 If $(a_n)_{n\ge1}, a_n \in \mathbb{R}_+^* = (0,\infty), n \in \mathbb{N}^*$ and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n \cdot \sqrt[n]{n!}} = a > 0$. Find:

$$\Omega(\mathrm{a}) = \lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

UP.477 If f is nonnegative function on $[0,1]$ and $f'(x) \ge 1$ then:

$$\int_0^x f^n(t)\,dt \ge x^{n-3}\left(\int_0^x f(t)\,dt\right)^{n-1}\ ;n\in\mathbb{N}, n\ge 2$$

Proposed by Florică Anastase-Romania

UP.478 If f is nonnegative function on $[0,1]$ and $f'(x) \ge 1$ then:

$$\int_0^x f^n(t)\,dt \ge x^{n-3}\left(\left(\int_0^x f(t)\,dt\right)^{n-1} + f^2(0)\left(\int_0^x f(t)\,dt\right)^{n-2}\right); n\in\mathbb{N}, n\ge 2$$

Proposed by Florică Anastase-Romania

UP.479 Let m_a, m_b, m_c be the lengths of the medians of a triangle ABC with circumradius R and inradius r. Let r_a, r_b, r_c be the exradii of the triangle.

Prove that: $72\frac{r^4}{R^3} \le \frac{m_a^2}{r_a} + \frac{m_b^2}{r_b} + \frac{m_c^2}{r_c} \le \frac{9}{8}\frac{R^4-8r^4}{r^3}$

Proposed by George Apostolopoulos-Greece

UP.480 If $(a_n)_{n\ge1}, (b_n)_{n\ge1}$ are sequence of real numbers with $a_n \ne a_{n+1}, b_n \ne b_{n+1}$ with $\lim_{n\to\infty} b_n = a \in \mathbb{R}$, $\lim_{n\to\infty} b_n = b \in \mathbb{R}$, $\lim_{n\to\infty}\big(n(a_{n+1}-a_n)\big) = c \in \mathbb{R}$,

$\lim_{n\to\infty}\big(n(b_{n+1}-b_n)\big) = d \in \mathbb{R}$ and $f, g: \mathbb{R}\to\mathbb{R}$ are differentiable functions with continuous derivative on $\mathbb{R}$, then find in terms of a, b, c, d:

$$\Omega = \lim_{n\to\infty}\Big(n\big(f(a_{n+1})g(b_{n+1}) - f(a_n)g(b_n)\big)\Big)$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

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NOTĂ: Pentru a publica probleme propuse, articole și note matematice în RMM puteți trimite materialele pe mailul: dansitaru63@yahoo.com