ROMANIAN MATHEMATICAL MAGAZINE

**Solve in $\mathbb{R}_+$ the system:**

$$\begin{cases} x^2+xy+y^2=(2x+y)\sqrt[3]{xz^2} \\ y^2+yz+z^2=(2y+z)\sqrt[3]{yx^2} \\ z^2+zx+x^2=(2z+x)\sqrt[3]{zy^2} \end{cases}$$

*Proposed by Mihaly Bencze, Neculai Stanciu-Romania*

*Solution 1 by Mohamed Amine Ben Ajiba-Morocco*

By AM − GM inequality, we have

$$3(x^2+xy+y^2)=(2x+y).3\sqrt[3]{xz^2}\le(2x+y)(x+2z).$$

Similarly, we get

$$3(y^2+yz+z^2)\le(2y+z)(y+2x) \text{ and } 3(z^2+zx+x^2)\le(2z+x)(z+2y).$$

Adding these inequality, we have

$$6(x^2+y^2+z^2)+3(xy+yz+zx)\le 2(x^2+y^2+z^2)+7(xy+yz+zx)$$

$$\Leftrightarrow 2(x-y)^2+2(y-z)^2+2(z-x)^2\le 0 \Rightarrow x-y=y-z=z-x=0.$$

So, the solution is $x=y=z$.

*Solution 2 by Mohamed Amine Ben Ajiba-Morocco*

Multiplying the three equations, we have

$$(x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2)=xyz(2x+y)(2y+z)(2z+x).$$

But by AM − GM inequality, we have

$$x^2+xy+y^2=(2x+y).\frac{x+2y}{3}+\frac{(x-y)^2}{3}\ge(2x+y)\sqrt[3]{xy^2},$$

with equality for $x=y$. Similarly, we have

$$y^2+yz+z^2\ge(2y+z)\sqrt[3]{yz^2} \text{ and } z^2+zx+x^2\ge(2z+x)\sqrt[3]{zx^2}.$$

Multiplying these inequalities, we have

$$(x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2)\ge xyz(2x+y)(2y+z)(2z+x),$$

with equality for $x=y=z$. So, the only solution of the system is $x=y=z$.

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