ROMANIAN MATHEMATICAL MAGAZINE

Solve for positive real numbers the following system:

$$\begin{cases} \dfrac{x^3}{(1+y)(1+z)} = \dfrac{6y-x-z-2}{8} \\ \dfrac{y^3}{(1+z)(1+x)} = \dfrac{6z-y-x-2}{8} \\ \dfrac{z^3}{(1+x)(1+y)} = \dfrac{6x-z-y-2}{8} \end{cases}$$

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Solution 1 by Pham Duc Nam-Vietnam

Sum 3 equations we get: $\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+z)(1+x)} + \frac{z^3}{(1+x)(1+y)} = \frac{4(x+y+z)-6}{8} = \frac{2(x+y+z)-3}{4}$ **(1)**

Now, we have: $\frac{x^3}{(1+y)(1+z)} + \frac{1+y}{8} + \frac{1+z}{8} \overset{AM-GM}{\geq} 3\sqrt[3]{\frac{x^3}{(1+y)(1+z)}\frac{1+y}{8}\frac{1+z}{8}} = \frac{3x}{4}$

Similarly, $\frac{y^3}{(1+z)(1+x)} + \frac{1+z}{8} + \frac{1+x}{8} \geq \frac{3y}{4}, \frac{z^3}{(1+x)(1+y)} + \frac{1+x}{8} + \frac{1+y}{8} \geq \frac{3z}{4}$

$$\Rightarrow \frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+z)(1+x)} + \frac{z^3}{(1+x)(1+y)} + \frac{1+x}{4} + \frac{1+y}{4} + \frac{1+z}{4} \geq$$
$$\geq \frac{3(x+y+z)}{4}$$

$$\Leftrightarrow \frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+z)(1+x)} + \frac{z^3}{(1+x)(1+y)} \geq$$
$$\geq \frac{3(x+y+z)}{4} - \frac{(x+y+z)}{4} - \frac{3}{4} = \frac{2(x+y+z)-3}{4} \quad \textbf{(2)}$$

From (1), (2): $\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+z)(1+x)} + \frac{z^3}{(1+x)(1+y)} = \frac{2(x+y+z)-3}{4} \Rightarrow x = y = z$

* Replace $x = y = z$ to the first equation (or any equation) we have: $\frac{x^3}{(1+x)^2} = \frac{2x-1}{4}$

$$\Leftrightarrow 4x^3 - (2x-1)(1+x)^2 = 0 \Leftrightarrow 2x^3 - 3x^2 + 1 = 0 \Leftrightarrow (2x+1)(x-1)^2 = 0 \overset{x\in\mathbb{R}^+}{\rightarrow}$$
$$x = 1 = y = z = 1$$

$\Rightarrow (x, y, z) = (1, 1, 1)$ is the unique solution of the given system of equations.

Solution 2 by Eric-Dimitrie Cismaru-Romania

Adding the three equations, we obtain $\sum_{cyc}\frac{x^3}{(1+y)(1+z)}=\frac{x+y+z}{2}-\frac{3}{4}$**. We prove that**

$$LHS \geq RHS$$

The triplets $\{x^3,y^3,z^3\}$ **and** $\left\{\frac{1}{(1+y)(1+z)},\frac{1}{(1+z)(1+x)},\frac{1}{(1+x)(1+y)}\right\}$ **are ordered the same. By Chebyshev's Inequality,**

$$LHS \geq \frac{x^3+y^3+z^3}{3}\cdot\sum_{cyc}\frac{1}{(1+y)(1+z)} \overset{Holder}{\geq} \left(\frac{x+y+z}{3}\right)^3\cdot\frac{x+y+z+3}{\prod(1+x)}=\frac{3\alpha^3(\alpha+1)}{\prod(1+x)},$$

where $\alpha=\frac{x+y+z}{3}$**. Denote** $a=x+1, b=y+1$ **and** $c=z+1$**. Then, by AM-GM,**

$$(a+b+c)^3 \geq \left(3\sqrt[3]{abc}\right)^3=27abc, \text{ or, in other words,}$$

$$\prod_{cyc}(1+x) \leq \frac{(a+b+c)^3}{27}=\frac{(x+y+z+3)^3}{27}=\left(\frac{x+y+z}{3}+1\right)^3=(\alpha+1)^3. \text{ Therefore, we get}$$

$$LHS \geq \frac{3\alpha^3(\alpha+1)}{(\alpha+1)^3}=\frac{3\alpha^3}{(\alpha+1)^2} \geq RHS=\frac{3\alpha}{2}-\frac{3}{4} \Leftrightarrow \frac{(\alpha-1)^2(2\alpha+1)}{(\alpha+1)^2} \geq 0 \Leftrightarrow LHS \geq RHS.$$

Since $RHS=LHS\geq RHS$**, equality holds in all the previous inequalities, which is,** $\alpha=1$ **and** $x=y=z$**, so** $S=\{1,1,1\}$ **is the only solution for the given system of equations.**