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PROBLEMS FOR JUNIORS

JP.526 Let $a,b,c\in(0,1)$. Prove that:

$$\frac{b+c}{1-a}+\frac{c+a}{1-b}+\frac{a+b}{1-c}\ge\frac{2\sqrt{ab}}{1-\sqrt{ab}}+\frac{2\sqrt{bc}}{1-\sqrt{bc}}+\frac{2\sqrt{ca}}{1-\sqrt{ca}}$$

Proposed by Marin Chirciu – Romania

Solution 1 by proposer

We prove: Lemma: Let $a,b\in(0,1)$. Prove that:

$$\frac{a}{1-b}+\frac{b}{1-a}\ge\frac{2\sqrt{ab}}{1-\sqrt{ab}}$$

Solution: Denoting $\sqrt{a}=x,\sqrt{b}=y$, we have $x,y\in(0,1)$ and we must show that $\frac{x^2}{1-y^2}+\frac{y^2}{1-x^2}\ge\frac{2xy}{1-xy}$, which follows from Bergstrom's inequality:

$$LHS=\frac{x^2}{1-y^2}+\frac{y^2}{1-x^2}\ge\frac{(x+y)^2}{(1-y^2)+(1-x^2)}=\frac{x^2+y^2+2xy}{2-x^2-y^2}\overset{(1)}{\ge}\frac{2xy}{1-xy}=RHS,$$ **where (1) $\Leftrightarrow$**

$\Leftrightarrow\frac{x^2+y^2+2xy}{2-x^2-y^2}\ge\frac{2xy}{1-xy}\Leftrightarrow(x-y)^2(xy+1)\ge0$, obviously with equality for $x=y$.

Let get back to the main problem. Using the Lemma we obtain:

$$\frac{a}{1-b}+\frac{b}{1-a}\ge\frac{2\sqrt{ab}}{1-\sqrt{ab}},\frac{b}{1-c}+\frac{c}{1-b}\ge\frac{2\sqrt{bc}}{1-\sqrt{bc}},\frac{c}{1-a}+\frac{a}{1-c}\ge\frac{2\sqrt{ca}}{1-\sqrt{ca}}$$

Adding the 3 above inequalities we obtain the conclusion.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM – GM inequality, we have

$$\frac{b}{1-a}+\frac{a}{1-b}\ge2\sqrt{\frac{ab}{(1-a)(1-b)}}=2\sqrt{\frac{ab}{\left(1-\sqrt{ab}\right)^2-\left(\sqrt{a}-\sqrt{b}\right)^2}}\ge\frac{2\sqrt{ab}}{1-\sqrt{ab}}.$$

Similarly, we have

$$\frac{c}{1-b}+\frac{b}{1-c}\ge\frac{2\sqrt{bc}}{1-\sqrt{bc}}\ \text{and}\ \frac{a}{1-c}+\frac{c}{1-a}\ge\frac{2\sqrt{ca}}{1-\sqrt{ca}}.$$

Adding these inequalities yields the desired result. Equality holds iff $a=b=c$.

JP.527 If $x,y,z>0$ with $x+y+z=1$ and $\lambda\ge 21$ then:

$$\frac{1}{x^3+y^3+z^3}+\frac{\lambda}{xy+yz+zx}\ge 3(\lambda+3)$$

Proposed by Marin Chirciu – Romania

Solution 1 by proposer

We use pqr method.

We denote: $p=x+y+z=1, q=xy+yz+xz, r=xyz.$

We have: $x^3+y^3+z^3=p^3-3pq+3r=1-3q+3r$

We write the inequality: $\frac{1}{1-3q+3r}+\frac{\lambda}{q}\ge 3(\lambda+3).$

Because $3r=3xyz=3xyz(x+y+z)\le(xy+yz+zx)^2=q^2$, it suffices to prove that:

$$\frac{1}{1-3q+q^2}+\frac{\lambda}{q}\ge 3(\lambda+3)\Leftrightarrow 3(\lambda+3)q^3-(10\lambda+27)q^2+(6\lambda+8)q-\lambda\le 0\Leftrightarrow$$

$$(3q-1)[(\lambda+3)q^2-(3\lambda+8)q+\lambda]\le 0$$

which follows from $q\le\frac{1}{3}$, true form $1=(x+y+z)^2\ge 3(xy+yz+zx)=3q.$

Equality holds if and only if $x=y=z=\frac{1}{3}$.

Remark: The best equality it's obtained for $\lambda=21.$

Case $\lambda=21$

If $x,y,z>0$ with $x+y+z=1$ then

$$\frac{1}{x^3+y^3+z^3}+\frac{21}{xy+yz+zx}\ge 72$$

Marin Chirciu

Solution

We use pqr method.

We denote: $p=x+y+z=1, q=xy+yz+xz, r=xyz.$

We have: $x^3+y^3+z^3=p^3-3pq+3r=1-3q+3r.$

The inequality can be written: $\frac{1}{1-3q+3r}+\frac{21}{q}\ge 72.$

Because $3r = 3xyz = 3xyz(x+y+z) \le (xy+yz+zx)^2 = q^2$, it suffices to prove that:

$$\frac{1}{1-3q+q^2} + \frac{21}{q} \ge 72 \Leftrightarrow 72q^3 - 237q^2 + 134q - 21 \le 0 \Leftrightarrow (3q-1)^2(8q-21) \le 0,$$

which follows from $q \le \frac{1}{3}$, true from $1 = (x+y+z)^2 \ge 3(xy+yz+zx) = 3q$.

Equality holds if and only if $x = y = z = \frac{1}{3}$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $p := x+y+z = 1,\ \ q := xy+yz+zx,\ \ r := xyz.$ Since

$$x^3+y^3+z^3 = p^3 - 3pq + 3r$$

then the given inequality can be rewritten as follows

$$\lambda\left(\frac{1}{q} - 3\right) \ge 9 - \frac{1}{1-3q+3r}.$$

Since $(x+y+z)^2 \ge 3(xy+yz+zx)$, then $q \le \frac{1}{3}$.

And since we have $(xy+yz+zx)^2 \ge 3xyz(x+y+z)$, then

$3r \le q^2$, so it suffices to prove that

$$21\left(\frac{1}{q} - 3\right) \ge 9 - \frac{1}{1-3q+q^2} \Leftrightarrow \frac{(1-3q)^2(21-8q)}{q(1-3q+q^2)} \ge 0,$$

which is true for $q \le \frac{1}{3}$. So the proof is complete. Equality holds iff $x = y = z = \frac{1}{3}$.

JP.528 If $a, b, c > 0$ such that $a+b+c = 3$ and $0 \le \lambda \le \frac{1}{2}$ then

$$\frac{1}{a^3+\lambda} + \frac{1}{b^3+\lambda} + \frac{1}{c^3+\lambda} \ge \frac{3}{\lambda+1}$$

Proposed by Marin Chirciu – Romania

Solution 1 by proposer

We prove: Lemma: If $x > 0$ and $0 \le \lambda \le \frac{1}{2}$ then

$$\frac{1}{x^3+\lambda} \ge \frac{\lambda+4-3x}{(\lambda+1)^2}$$

Proof: We use Tangent Line Method.We consider the function $f: (0,\infty) \to \mathbb{R}, f(x) = \frac{1}{x^3+\lambda}$

We have $f(1) = \frac{1}{\lambda+1}$. Tangent's equation in point $x_0 = 1$ is $y - f(x_0) = f'(x_0)(x - x_0)$.

We have $f'(x) = \frac{-3x^2}{(x^3+\lambda)^2}, f'(1) = \frac{-3}{(\lambda+1)^2}$.

Tangent's equation in point $x_0=1$ is $y-\frac{1}{\lambda+1}=\frac{-3}{(\lambda+1)^2}(x-1)$.

We obtain the tangent's equation $y=\frac{\lambda+4}{(\lambda+1)^2}-\frac{3x}{(\lambda+1)^2}$. We prove that:

$f(x)=\frac{1}{x^3+\lambda}\geq\frac{\lambda+4}{(\lambda+1)^2}-\frac{3x}{(\lambda+1)^2}$. We have:

$\frac{1}{x^3+\lambda}\geq\frac{\lambda+4-3x}{(\lambda+1)^2}\Leftrightarrow(x-1)^2(3x^2+(2-\lambda)x+1-2\lambda)\geq 0$, which follows from the condition from the hypothesis $0\leq\lambda\leq\frac{1}{2}$. Let's get back to the main problem.

Using the Lemma we obtain:

$$LHS=\sum\frac{1}{a^3+\lambda}\geq\sum\frac{\lambda+4-3a}{(\lambda+1)^2}=\frac{3(\lambda+4)-3\sum a}{(\lambda+1)^2}=\frac{3(\lambda+4)-3\cdot 3}{(\lambda+1)^2}=\frac{3}{\lambda+1}=RHS.$$

Equality holds if and only if $(a,b,c)=(1,1,1)$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We will first prove the lemma that for all $a\in(0,1)$ and $\lambda\in\left[0,\frac{1}{2}\right]$, we have

$$\frac{1}{a^3+\lambda}\geq\frac{-3a+\lambda+4}{(\lambda+1)^2}.$$

The inequality is equivalent to

$$\frac{3a^4-(\lambda+4)a^3+3\lambda a+1-2\lambda}{(a^3+\lambda)(\lambda+1)^2}\geq 0\ \text{ or }\ \frac{(a-1)^2[3a^2+(2-\lambda)a+1-2\lambda]}{(a^3+\lambda)(\lambda+1)^2}\geq 0,$$

which is true and the proof of lemma is complete. Using this lemma, we obtain

$$\frac{1}{a^3+\lambda}+\frac{1}{b^3+\lambda}+\frac{1}{c^3+\lambda}\geq\frac{-3a+\lambda+4}{(\lambda+1)^2}+\frac{-3b+\lambda+4}{(\lambda+1)^2}+\frac{-3c+\lambda+4}{(\lambda+1)^2}$$
$$=\frac{-3(a+b+c)+3(\lambda+4)}{(\lambda+1)^2}=\frac{3}{\lambda+1},$$

as desired. Equality holds iff $a=b=c=1$.

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\frac{1}{a^3+\lambda}-\frac{1}{1+\lambda}-\frac{3(1-a)}{(1+\lambda)^2}=\frac{(1-a)(1+a+a^2)}{(a^3+\lambda)(1+\lambda)}-\frac{3(1-a)}{(1+\lambda)^2}$$
$$=\frac{1-a}{1+\lambda}\cdot\left(\frac{1+a+a^2}{a^3+\lambda}-\frac{3}{1+\lambda}\right)=\frac{1-a}{1+\lambda}\cdot\frac{1+\lambda+a+a\lambda+a^2+a^2\lambda-3a^3-3\lambda}{(a^3+\lambda)(1+\lambda)}$$
$$=\frac{1-a}{1+\lambda}\cdot\frac{(1-a)(1+a+a^2)+a(1-a)(1+a)+a^2(1-a)+\lambda(a^2+a-2)}{(a^3+\lambda)(1+\lambda)}$$
$$=\frac{1-a}{1+\lambda}\cdot\frac{(1-a)(1+a+a^2)+a(1-a)(1+a)+a^2(1-a)-\lambda(1-a)(a+2)}{(a^3+\lambda)(1+\lambda)}$$

$$=\frac{(1-a)^2}{1+\lambda}\cdot\frac{1+a+a^2+a+a^2+a^2-\lambda(a+2)}{(a^3+\lambda)(1+\lambda)}\overset{0\le\lambda\le\frac{1}{2}}{\ge}$$

$$=\frac{(1-a)^2}{1+\lambda}\cdot\frac{1+2a+3a^2-\frac{1}{2}(a+2)}{(a^3+\lambda)(1+\lambda)}=\frac{(1-a)^2}{1+\lambda}\cdot\frac{\frac{3a}{2}+3a^2}{(a^3+\lambda)(1+\lambda)}\ge 0$$

$\because a>0$ and $0\le\lambda\le\frac{1}{2}\therefore\frac{1}{a^3+\lambda}\ge\frac{1}{1+\lambda}+\frac{3(1-a)}{(1+\lambda)^2}$ and analogs

$$\Rightarrow\frac{1}{a^3+\lambda}+\frac{1}{b^3+\lambda}+\frac{1}{c^3+\lambda}\ge\frac{3}{\lambda+1}+\frac{3}{(1+\lambda)^2}\cdot\left(3-\sum_{cyc}a\right)\overset{a+b+c=3}{=}\frac{3}{\lambda+1}$$

$\therefore\frac{1}{a^3+\lambda}+\frac{1}{b^3+\lambda}+\frac{1}{c^3+\lambda}\ge\frac{3}{\lambda+1}\ \forall a,b,c>0$ such that : $a+b+c=3$ and $0\le\lambda\le\frac{1}{2}$, " = " iff $a=b=c=1$ (QED)

JP.529 If $a,b,c>0; x\in\mathbb{R}$ then:

$$\frac{a}{(b\sin^2x+c\cos^2x)^3}+\frac{b}{(c\sin^2x+b\cos^2x)^3}+\frac{c}{(a\sin^2x+c\cos^2x)^3}\ge\frac{27}{(a+b+c)^2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$\sum_{cyc}\frac{a}{b\sin^2x+c\cos^2x}=\sum_{cyc}\frac{a^2}{ab\sin^2x+ac\cos^2x}\ge$$

$$\overset{BERGSTROM}{\ge}\frac{(a+b+c)^2}{(ab+bc+ca)(\sin^2x+\cos^2x)}=\frac{(a+b+c)^2}{ab+bc+ca}\ge 3$$

Because: $(a+b+c)^2\ge 3(ab+bc+ca)$

$$a^2+b^2+c^2\ge ab+bc+ca,\ (a-b)^2+(b-c)^2+(c-a)^2\ge 0$$

Hence:

$$\sum_{cyc}\frac{a}{b\sin^2x+c\cos^2x}\ge 3\qquad(1)$$

$$\sum_{cyc}\frac{a}{(b\sin^2x+c\cos^2x)^3}=\sum_{cyc}\frac{\left(\frac{a}{b\sin^2x+c\cos^2x}\right)^3}{a^2}\ge$$

$$\overset{RADON}{\ge}\frac{1}{(a+b+c)^2}\cdot\left(\sum_{cyc}\frac{a}{b\sin^2x+c\cos^2x}\right)^3\overset{(1)}{\ge}\frac{27}{(a+b+c)^2}$$

Equality holds for: $a=b=c$.

Solution 2 by Marin Chirciu-Romania

$$\sum\frac{a}{(b\sin^2 x+c\cos^2 x)^3}\ge\frac{27}{(a+b+c)^2}$$

$$LHS=\sum\frac{a}{(b\sin^2 x+c\cos^2 x)^3}=\sum\frac{\left(\frac{a}{b\sin^2 x+c\cos^2 x}\right)^3}{a^2}\overset{Radon}{\ge}$$

$$\overset{Radon}{\ge}\frac{\left(\sum\frac{a}{b\sin^2 x+c\cos^2 x}\right)^3}{(\sum a)^2}\overset{(1)}{\ge}\frac{3^3}{(\sum a)^2}=\frac{27}{(a+b+c)^2}=RHS,$$

where (1) $\Leftrightarrow\sum\frac{a}{b\sin^2 x+c\cos^2 x}\ge 3$, which follows from

$$\sum\frac{a}{b\sin^2 x+c\cos^2 x}=\sum\frac{a^2}{ab\sin^2 x+ac\cos^2 x}\overset{CS}{\ge}\frac{(\sum a)^2}{\sum(ab\sin^2 x+ac\cos^2 x)}=$$

$=\frac{(\sum a)^2}{\sum bc(\sin^2 x+\cos^2 x)}=\frac{(\sum a)^2}{\sum bc}\overset{SOS}{\ge}3$. **Equality holds if and only if $a=b=c$.**

Remark: The problem can be developed.

If $a,b,c>0,x\in\mathbb{R}$ and $n\in\mathbb{N}$ then

$$\sum\frac{a}{(b\sin^2 x+c\cos^2 x)^{n+1}}\ge\frac{3^{n+1}}{(a+b+c)^n}$$

Marin Chirciu – Romania

Solution:

$$LHS=\sum\frac{a}{(b\sin^2 x+c\cos^2 x)^{n+1}}=\sum\frac{\left(\frac{a}{b\sin^2 x+c\cos^2 x}\right)^{n+1}}{a^n}\overset{Radon}{\ge}$$

$$\overset{Radon}{\ge}\frac{\left(\sum\frac{a}{b\sin^2 x+c\cos^2 x}\right)^{n+1}}{(\sum a)^n}\overset{(1)}{\ge}\frac{3^{n+1}}{(\sum a)^n}=\frac{3^{n+1}}{(a+b+c)^n}=RHS,$$

where (1) $\Leftrightarrow\sum\frac{a}{b\sin^2 x+c\cos^2 x}\ge 3$, which follows from

$$\sum\frac{a}{b\sin^2 x+c\cos^2 x}=\sum\frac{a^2}{ab\sin^2 x+ac\cos^2 x}\overset{CS}{\ge}\frac{(\sum a)^2}{\sum(ab\sin^2 x+ac\cos^2 x)}=$$

$$=\frac{(\sum a)^2}{\sum bc\,(\sin^2 x+\cos^2 x)}=\frac{(\sum a)^2}{\sum bc}\overset{SOS}{\ge}3.$$

Equality holds if and only if $a=b=c$. Note: For $n=2$ we obtain Problem JP.529 from RMM, Number 36, Spring 2025, propose by Daniel Sitaru, Romania

Solution 3 by Tapas Das – India

$$\frac{a}{(b\sin^2 x+c\cos^2 x)^3}+\frac{b}{(c\sin^2 x+a\cos^2 x)^3}+\frac{c}{(a\sin^2 x+b\cos^2 x)^3}=$$

$$=\frac{a^4}{(ab\sin^2 x+ac\cos^2 x)^3}+\frac{b^4}{(bc\sin^2 x+ab\cos^2 x)^3}+\frac{c^4}{(ac\sin^2 x+bc\cos^2 x)^3}$$

$$\overset{Radon}{\geq}\frac{(a+b+c)^4}{[(\sin^2 x+\cos^2 x)(ab+bc+ca)]^3}\geq\frac{(a+b+c)^4}{\left[\frac{(a+b+c)^2}{3}\right]^3}=\frac{27}{(a+b+c)^2}$$

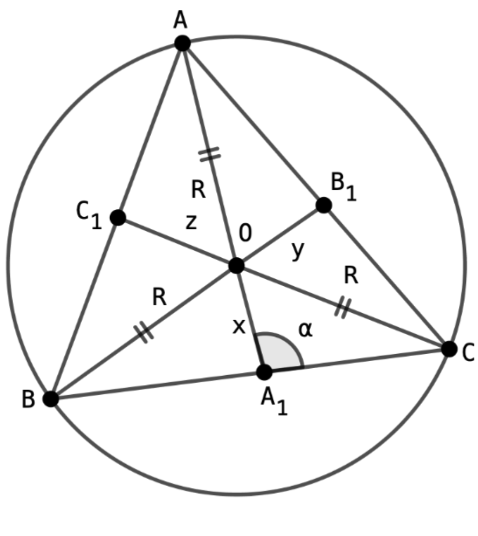
[Note: $\therefore (2x)^2\geq 3\sum xy$]

JP.530 In $\Delta ABC, O-$ circumcenter. A_1,B_1,C_1 are the intersection points of AO,BO,CO with BC,AC and AB respectively. R_1,R_2 and R_3 are circumradii of $\Delta BOC,\Delta AOC$ and ΔAOB respectively. Show that

$$R\left(\frac{1}{OA_1}+\frac{1}{OB_1}+\frac{1}{OC_1}\right)+3=\frac{4F}{R^2}\left(\frac{R_1}{BC}+\frac{R_2}{AC}+\frac{R_3}{AB}\right)$$

Proposed by Ertan Yildirim-Turkiye

Solution 1 by proposer



$$\frac{F}{[BOC]}=\frac{\frac{1}{2}(R+x)\cdot a\cdot\sin\alpha}{\frac{1}{2}\cdot x\cdot a\cdot\sin\alpha}=\frac{R+x}{x}=\frac{R}{x}+1$$

$$\frac{\frac{abc}{4R}}{R^2\cdot\frac{a}{4R_1}}=\frac{abc\cdot R_1}{a\cdot R^3}=\frac{R}{x}+1\Rightarrow\frac{4F\cdot R\cdot R_1}{a\cdot R^3}=\frac{4F}{R^2}\cdot\frac{R_1}{a}=\frac{R}{x}+1$$

Similarly $\frac{4F}{R^2}\cdot\frac{R_2}{b}=\frac{R}{y}+1,\frac{4F}{R^2}\cdot\frac{R_3}{c}=\frac{R}{z}+1$

$$\frac{4F}{R^2}\cdot\left(\frac{R_1}{a}+\frac{R_2}{b}+\frac{R_3}{c}\right)=3+R\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

$$\frac{4F}{R^2}\cdot\left(\frac{R_1}{BC}+\frac{R_2}{AC}+\frac{R_3}{AB}\right)=3+R\left(\frac{1}{OA_1}+\frac{1}{OB_1}+\frac{1}{OC_1}\right)$$

Solution 2 by Marin Chirciu – Romania

We evaluate the left side member.

Let be D – the leg of the height from A and M – the left of the perpendicular from O on BC, d_a – the distance from O to BC.

We have $\Delta ADC\sim\Delta OMA_1\Rightarrow\frac{AD}{OM}=\frac{AA_1}{OA_1}\Leftrightarrow\frac{h_a}{d_a}=\frac{R+OA_1}{OA_1}\Leftrightarrow\frac{h_a}{d_a}=\frac{R}{OA_1}=$

$$=\frac{R}{OA_1}+1\Leftrightarrow\frac{R}{OA_1}=\frac{h_a}{d_a}-1$$

We obtain:

$$LHS=R\left(\frac{1}{OA_1}+\frac{1}{OB_1}+\frac{1}{OC_1}\right)+3=\sum\left(\frac{h_a}{d_a}-1\right)=3=\sum\frac{h_a}{d_a}=\sum\frac{ah_a}{ad_a}=\sum\frac{2F}{ad_a}=2F\sum\frac{1}{ad_a}\quad(1)$$

We evaluate the right hand member.

$$R_1=\frac{OB\cdot OC\cdot BC}{4[BOC]}=\frac{R\cdot R\cdot a}{4\cdot\frac{a\cdot d_a}{2}}=\frac{R^2}{2d_a}\Rightarrow\frac{R_1}{BC}=\frac{\frac{R^2}{2d_a}}{a}=\frac{R^2}{2ad_a}$$

$$RHS=\frac{4F}{R^2}\left(\frac{R_1}{BC}+\frac{R_2}{CA}+\frac{R_3}{AB}\right)=\frac{4F}{R^2}\sum\frac{R_1}{BC}=\frac{4F}{R^2}\sum\frac{R^2}{2ad_a}=2F\sum\frac{1}{ad_a}\qquad(2)$$

From (1) and (2) we deduce the conclusion.

JP.531 If $a,b,c>0$ and $abc=1$ then:

$$\left(\frac{a}{2}+\frac{b}{c}\right)^3+\left(\frac{b}{2}+\frac{c}{a}\right)^3+\left(\frac{c}{2}+\frac{a}{b}\right)^3>3\sqrt[4]{18}$$

Proposed by Khaled Abd Imouti-Syria

Solution 1 by proposer

$$\left(a+\frac{2b}{c}\right)^3=a^3+3a^2\cdot\frac{3b}{c}+3a\cdot\frac{4b^2}{c^2}+\frac{8b^3}{c^3}$$

$$\left(a+\frac{2b}{c}\right)^3=a^3+6\frac{a^2b}{c}+12\frac{ab^2}{c^2}+\frac{8b^3}{c^3}$$

$$\left(a+\frac{2b}{c}\right)^3>4\sqrt[4]{\frac{6\cdot 12\cdot 8\cdot a^6\cdot b^6}{c^6}},\left(a+\frac{2b}{c}\right)^3>4\sqrt[4]{\frac{2^4\cdot 18\cdot a^8\cdot b^8\cdot c^2}{c^8a^2b^2}}$$

$$\left(a+\frac{2b}{c}\right)^3>4\cdot 2\sqrt[4]{\frac{18\,a^8b^8c^2}{c^8\cdot a^2\cdot b^2}},\left(a+\frac{2b}{c}\right)^3>8\cdot\frac{a^2b^2}{c^2}\sqrt[4]{\frac{18c^2}{a^2b^2}}$$

$$abc=1\Rightarrow c=\frac{1}{ab},\qquad c^2=\frac{1}{a^2b^2},\qquad \frac{c^2}{a^2b^2}=\frac{1}{a^4b^4}$$

$$\left(a+\frac{2b}{c}\right)^3>8\frac{a^2b^2}{c^2}\sqrt[4]{\frac{18}{a^4b^4}},\left(a+\frac{2b}{c}\right)^3>8\cdot\frac{a^2b^2}{c^2}\cdot\frac{\sqrt[4]{18}}{ab}$$

$$\left(a+\frac{2b}{c}\right)^3>8\sqrt[4]{18}\cdot\frac{ab}{c^2}$$

$$\left(a+\frac{2b}{c}\right)^3+\left(b+\frac{2c}{a}\right)^3+\left(c+\frac{2a}{b}\right)^3>8\sqrt[4]{18}\left[\frac{ab}{c^2}+\frac{bc}{a^2}+\frac{ca}{b^2}\right]$$

$$\geq 8\sqrt[4]{18}\cdot 3\cdot\sqrt[3]{\frac{a^2b^2c^2}{a^2b^2c^2}}$$

$$\left(a+\frac{2b}{c}\right)^3+\left(b+\frac{2c}{a}\right)^3+\left(c+\frac{2a}{b}\right)^3>24\sqrt[4]{18}$$

$$\left(\frac{a}{2}+\frac{b}{c}\right)^3+\left(\frac{b}{2}+\frac{c}{a}\right)^3+\left(\frac{c}{2}+\frac{a}{b}\right)^3>3\sqrt[4]{18}$$

Solution 2 by Marin Chirciu – Romania

$$\sum\left(\frac{a}{2}+\frac{b}{c}\right)^3\ge 3\sqrt[4]{18}$$

$$LHS=\sum\left(\frac{a}{2}+\frac{b}{c}\right)^3\overset{Holder}{\ge}\frac{\left[\sum\left(\frac{a}{b}+\frac{b}{c}\right)\right]^3}{9}=$$

$$=\frac{\left(\frac{a+b+c}{2}+\frac{b}{c}+\frac{c}{a}+\frac{a}{b}\right)^3}{9}\overset{AM-GM}{\ge}$$

$$\overset{AM-GM}{\ge}\frac{\left(\frac{3\sqrt[3]{abc}}{2}+3\sqrt[3]{\frac{b}{c}\cdot\frac{c}{a}\cdot\frac{a}{b}}\right)^3}{9}=\frac{\left(\frac{3}{2}+3\right)^3}{9}=\frac{\left(\frac{9}{2}\right)^3}{9}=\frac{9^2}{2^3}=$$

$$=\frac{81}{8}>3\sqrt[4]{18}=RHS$$

Remark: The problem can be developed.

If $a,b,c>0, abc=1$ and $n\in\mathbb{N}$ then:

$$\sum\left(\frac{a}{2}+\frac{b}{c}\right)^n\ge 3\left(\frac{3}{2}\right)^n$$

Marin Chirciu – Romania

Solution:

$$LHS=\sum\left(\frac{a}{2}+\frac{b}{c}\right)^n\overset{Holder}{\ge}\frac{\left[\sum\left(\frac{a}{2}+\frac{b}{c}\right)\right]^n}{3^{n-1}}=$$

$$=\frac{\left(\frac{a+b+c}{2}+\frac{b}{c}+\frac{c}{a}+\frac{a}{b}\right)^n}{3^{n-1}}\overset{AM-GM}{\ge}$$

$$\overset{AM-GM}{\ge}\frac{\left(\frac{3\sqrt[3]{abc}}{2}+3\sqrt[3]{\frac{b}{c}\cdot\frac{c}{a}\cdot\frac{a}{b}}\right)^n}{3^{n-1}}=\frac{\left(\frac{3}{2}+3\right)^n}{3^{n-1}}=\frac{\left(\frac{9}{2}\right)^n}{3^{n-1}}=3\left(\frac{3}{2}\right)^n=RHS$$

Equality holds if and only if $a=b=c$.

Note: For $n=3$ we obtain Problem JP.531 from RMM, Number 36, Spring 2025, proposed by Khaled Abd Imouti, Syria.

Again the problem can be developed.

If $a,b,c>0, abc=1$ and $n\in\mathbb{N}, \lambda>0$ then:

$$\sum\left(\frac{a}{\lambda}+\frac{b}{c}\right)^n\ge 3\left(1+\frac{1}{\lambda}\right)^n$$

Marin Chirciu – Romania

Solution:

$$LHS=\sum\left(\frac{a}{\lambda}+\frac{b}{c}\right)^n\overset{Holder}{\ge}\frac{\left[\sum\left(\frac{a}{\lambda}+\frac{b}{c}\right)\right]^n}{3^{n-1}}=\frac{\left(\frac{a+b+c}{\lambda}+\frac{b}{c}+\frac{c}{a}+\frac{a}{b}\right)^n}{3^{n-1}}\overset{AM-GM}{\ge}$$

$$\overset{AM-GM}{\ge}\frac{\left(\frac{3\sqrt[3]{abc}}{\lambda}+3\sqrt[3]{\frac{b}{c}\cdot\frac{c}{a}\cdot\frac{a}{b}}\right)^n}{3^{n-1}}=\frac{\left(\frac{3}{\lambda}+3\right)^n}{3^{n-1}}=\frac{\left(\frac{3\lambda+3}{\lambda}\right)^n}{3^{n-1}}=$$

$$=3\left(\frac{3\lambda+3}{3\lambda}\right)^n=3\left(\frac{\lambda+1}{\lambda}\right)^n=RHS$$

Equality holds if and only if $a=b=c$.

Note: For $n=3$ and $\lambda=2$ we obtain Problem JP.531 from RMM, Number 36, Spring 2025, proposed by Khaled Abd Imouti, Syria

JP.532 In any $\Delta ABC, I-$ incenter, $r-$ radii, $R-$ circumradii, $s-$ semiperimeter, the following relationship holds:

$$AB+BI+CI\le 2(R+r)$$

Proposed by Marian Ursărescu – Romania

Solution 1 by proposer

Firstly, we prove that

$$\frac{AI^2}{bc}+\frac{BC^2}{ca}+\frac{CI^2}{ab}=1\quad (1)$$

$$\sum_{cyc}\frac{AI^2}{bc}=\frac{1}{abc}\cdot\sum_{cyc}aAI^2=\frac{1}{abc}\sum_{cyc}a\cdot\frac{r^2}{\sin^2\frac{A}{2}}=\frac{r^2}{abc}\cdot\sum_{cyc}\frac{a}{\sin^2\frac{A}{2}}$$

But, it is well-known that in any triangle ABC holds:

$$\sum_{cyc}\frac{a}{\sin^2\frac{A}{2}}=\frac{4Rs}{r}$$

Hence,

$$\sum_{cyc}\frac{AI^2}{bc}=\frac{4Rrs}{abc}=\frac{4RF}{abc}=1\Rightarrow \text{(1) is true.}$$

Now, using Cauchy-Schwarz's inequality, we have:

$$\left(\frac{AI^2}{bc}+\frac{BC^2}{ca}+\frac{CI^2}{ab}\right)(ab+bc+ca)\geq(AI+BI+CI)^2\quad (2)$$

From (1) and (2), it follows that

$$(AI+BI+CI)^2\leq(ab+bc+ca)\quad (3)$$

But, $ab+bc+ca=s^2+r^2+4Rr$ (4)

From (3) and (4) we get: $(AI+BI+CI)^2\leq s^2+r^2+4Rr$ (5) and

$$s^2\leq 4R^2+4Rr+3r^2 \text{ (Gerretsen)}\quad (6)$$

From (5) and (6), it follows that:

$$(AI+BI+CI)^2\leq 4R^2+8Rr+4r^2\Leftrightarrow$$

$$(AI+BI+CI)^2\leq 4R^2+8Rr+4r^2\Leftrightarrow AI+BI+CI\leq 2(R+r)$$

Solution 2 by Tapas Das-India

$$AI+BI+CI=r\csc\frac{A}{2}+r\csc\frac{B}{2}+r\csc\frac{C}{2}=$$

$$=r\sqrt{\frac{bc}{(s-b)(s-c)}}+r\sqrt{\frac{ca}{(s-c)(s-a)}}+r\sqrt{\frac{ab}{(s-a)(s-b)}}$$

$$\overset{CBS}{\leq} r\sqrt{\left(\sum bc\right)\cdot\sum\frac{1}{(s-b)(s-c)}}$$

$$=r\sqrt{(s^2+r^2+4Rr)\cdot\frac{s}{(s-a)(s-b)(s-c)}}$$

$$\overset{Gerretsen's}{\leq} r\sqrt{(4R^2+4Rr+3r^2+r^2+4Rr)\frac{1}{r^2}}=\sqrt{4(R^2+2Rr+r^2)}=2(R+r)$$

Solution 3 by Marin Chirciu – Romania

$$LHS=\sum AI=\sum\frac{r}{\sin\frac{A}{2}}\overset{(1)}{\leq} r\cdot 2\left(\frac{R}{r}+1\right)=2(R+r)=RHS,$$

where (1) $\Leftrightarrow \sum\frac{1}{\sin\frac{A}{2}}\leq 2\left(\frac{R}{r}+1\right)$, which follows from:

$$\sum\frac{1}{\sin\frac{A}{2}}=\sum\frac{1}{\sqrt{\frac{(s-b)(s-c)}{bc}}}=\sum\frac{\sqrt{bc}}{\sqrt{(s-b)(s-c)}}\overset{CBS}{\le}$$

$$\overset{CBS}{\le}\sqrt{\sum bc\sum\frac{1}{(s-b)(s-c)}}=$$

$$=\sqrt{(s^2+r^2+4Rr)\frac{1}{r^2}}=\frac{1}{r}\sqrt{s^2+r^2+4Rr}\overset{Gerretsen}{\le}$$

$$\overset{Gerretsen}{\le}\frac{1}{r}\sqrt{4R^2+4Rr+3r^2+r^2+4Rr}=$$

$$=\frac{1}{r}\sqrt{4R^2+8Rr+4r^2}=\frac{2}{r}\sqrt{R^2+2Rr+r^2}=\frac{2}{r}(R+r)=2\left(\frac{R}{r}+1\right)$$

Equality holds if and only if the triangle is equilateral.

JP.533 Let be the triangle ABC with $AD, BE, CF -$ altitudes and $H -$ orthocenter. Prove that:

$$\frac{HA}{HD}+\frac{HB}{HE}+\frac{HC}{HF}\ge 2\left(\left(\frac{R}{r}\right)^2-1\right)$$

Proposed by Marian Ursărescu – Romania

Solution 1 by proposer

In any triangle ABC is well-known that:

$$\frac{HA}{HD}+\frac{HB}{HE}+\frac{HC}{HF}=\sum\nolimits_{cyc}\tan A\tan B-3 \qquad (1)$$

and

$$\sum_{cyc}\tan A\tan B=\frac{s^2-r^2-4Rr}{s^2-(2R+r)^2}-3=$$

$$=\frac{s^2-r^2-4Rr-3s^2+12R^2+12Rr+3r^2}{s^2-(2R+r)^2}=\frac{12R^2+8Rr+2r^2-2r^2}{s^2-(2R+r)^2} \quad (3)$$

But $s^2\le 4R^2+4Rr+3r^2$ (Gerretsen) (4)

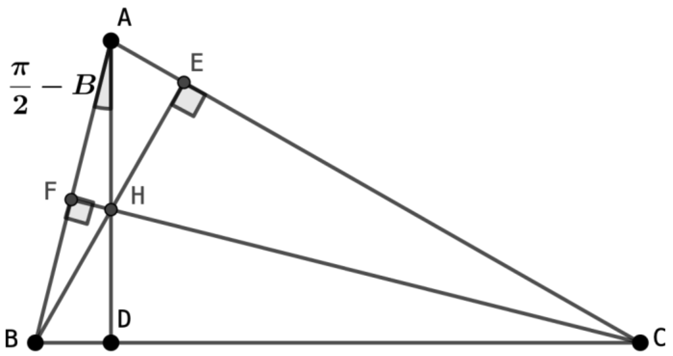
From (3) and (4), we get:

$$\frac{HA}{HD}+\frac{HB}{HE}+\frac{HC}{HF}\ge\frac{12R^2+8Rr+2r^2-8R^2-8Rr-6r^2}{4R^2+4Rr+3r^2-4R^2-4Rr-r^2}$$

Hence,

$$\frac{HA}{HD}+\frac{HB}{HE}+\frac{HC}{HF}\geq\frac{4R^2-4r^2}{2r^2}=\left(\left(\frac{R}{r}\right)^2-1\right)$$

Solution 2 by Tapas Das – India



$$\text{In } \Delta ADB, \angle BAD=\frac{\pi}{2}-B$$

$$\text{In } \Delta AFC, AF=b\cos A$$

$$\text{In } \Delta AFH, \cos\left(\frac{\pi}{2}-B\right)=\frac{AF}{AH}=\frac{b\cos A}{AH}\Rightarrow AH=\frac{b\cos A}{\cos\left(\frac{\pi}{2}-B\right)}=\frac{b\cos A}{\sin B}$$

$$AH=2R\cos A \text{ analog}$$

$$\text{In } \Delta AFH: \tan\left(\frac{\pi}{2}-B\right)=\frac{FH}{AF}=\frac{FH}{b\cos A}$$

$$FH=\frac{b\cos A\cos B}{\sin B}=2R\cos A\cos B \quad \text{(analog)}$$

Now,

$$\frac{HA}{HD}+\frac{HB}{HE}+\frac{HC}{HF}=\frac{2R\cos A}{2R\cos B\cos C}+\frac{2R\cos B}{2R\cos A\cos C}+\frac{2R\cos C}{2R\cos A\cos B}=$$

$$=\frac{\sum\cos^2 A}{\cos A\cos B\cos C}=\frac{(\sum\cos A)^2-2\sum\cos A\cos B}{\cos A\cos B\cos C}$$

$$=\frac{\left(1+\frac{r}{R}\right)^2-2\frac{s^2+r^2-4R^2}{4R^2}}{\frac{s^2-(2R+r)^2}{4R^2}}=\frac{4(R+r)^2-2(s^2+r^2-4R^2)}{s^2-(2R+r)^2}=$$

$$=\frac{4(R^2+2Rr+r^2)-2s^2-2r^2+8R^2}{s^2-(2R+r)^2}=\frac{12R^2+8Rr+2r^2-2s^2}{s^2-(4R^2+4Rr+r^2)}\geq$$

$$\overset{Gerretsen's}{\geq}\frac{12R^2+8Rr+2r^2-2(4R^2+4Rr+3r^2)}{4R^2+4Rr+3r^2-4R^2-4Rr-r^2}$$

$$=\frac{4R^2-4r^2}{2r^2}=2\left(\frac{R}{r}\right)^2-2=2\left[\left(\frac{R}{r}\right)^2-1\right]$$

Solution 3 by Marin Chirciu – Romania

In acute ΔABC we have $HA=2R\cos A$ and $HD=2R\cos B\cos C$.

We obtain:

$$LHS=\sum\frac{HA}{HD}=\sum\frac{2R\cos A}{2R\cos B\cos C}\sum\frac{\cos A}{\cos B\cos C}=\frac{\sum\cos^2 A}{\prod\cos A}=$$

$$=\frac{\frac{6R^2+4Rr+r^2-s^2}{2R^2}}{\frac{s^2-(2R+r)^2}{4R^2}}=\frac{2(6R^2+4Rr+r^2-s^2)}{s^2-(2R+r)^2}\overset{(1)}{\geq}2\left(\left(\frac{R}{r}\right)^2-1\right)=RHS$$

where (1) $\Leftrightarrow\frac{2(6R^2+4Rr+r^2-s^2)}{s^2-(2R+r)^2}\geq 2\left(\left(\frac{R}{r}\right)^2-1\right)\Leftrightarrow$

$$\Leftrightarrow\frac{(6R^2+4Rr+r^2-s^2)}{s^2-(2R+r)^2}\geq\frac{R^2-r^2}{r^2}\Leftrightarrow$$

$$\Leftrightarrow r^2(6R^2+4Rr+r^2-s^2)\geq(R^2-r^2)[s^2-(2R+r)^2]\Leftrightarrow$$

$$\Leftrightarrow R^2s^2\leq(R^2-r^2)(2R+r)^2+r^2(6R^2+4Rr+r^2)$$

which follows from Gerretsen's inequality $s^2\leq 4R^2+4Rr+3r^2$.

It remains to prove that:

$$R^2(4R^2+4Rr+3r^2)\leq(R^2-r^2)(2R+r)^2+r^2(6R^2+4Rr+r^2)\Leftrightarrow$$

$$\Leftrightarrow 3R^2r^2\leq 3R^2r^2\text{, obviously.}$$

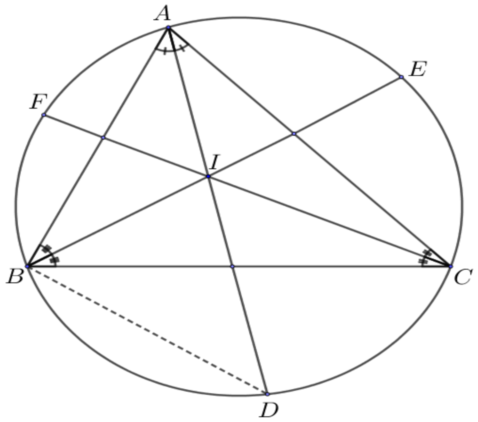
Equality holds if and only if the triangle is equilateral.

JP.534 In $\Delta ABC, I-$ incenter and D,E,F the points of contact of the cevians AI,BI,CI with the circumcircle, then the following relationship holds:

$$ID+IE+IF\leq\frac{2(R^2-Rr+r^2)}{r}$$

Proposed by Marian Ursărescu – Romania

Solution 1 by proposer



In ΔIBC, from Law of sines, it follows:

$$\frac{ID}{\sin(IBD)} = \frac{IB}{\sin(IDB)} \quad (1)$$

But:

$$\sin(IBD) = \sin\left(\frac{B}{2}+\frac{A}{2}\right) = \sin\left(\frac{\pi}{2}-\frac{C}{2}\right) = \cos\frac{C}{2} \quad (2)$$

$$\sin(IDB) = \sin C \quad (3)$$

From (1), (2) and (3), we get:

$$ID = \frac{ID\cdot\cos\frac{C}{2}}{\sin C} = \frac{BI}{2\sin\frac{C}{2}}; \text{ and analogs } (4)$$

By adding, we have:

$$ID + IE + IF = \frac{1}{2}\left(\frac{BI}{\sin\frac{C}{2}} + \frac{CI}{\sin\frac{A}{2}} + \frac{AI}{\sin\frac{B}{2}}\right)$$

$$(ID + IE + IF)^2 = \frac{1}{4}\left(\frac{BI}{\sin\frac{C}{2}} + \frac{CI}{\sin\frac{A}{2}} + \frac{AI}{\sin\frac{B}{2}}\right)^2 \quad (5)$$

$$\left(\frac{BI}{\sin\frac{C}{2}} + \frac{CI}{\sin\frac{A}{2}} + \frac{AI}{\sin\frac{B}{2}}\right) \overset{C-S}{\leq} (AI^2 + BI^2 + CI^2)\left(\frac{1}{\sin^2\frac{A}{2}} + \frac{1}{\sin^2\frac{B}{2}} + \frac{1}{\sin^2\frac{C}{2}}\right) =$$

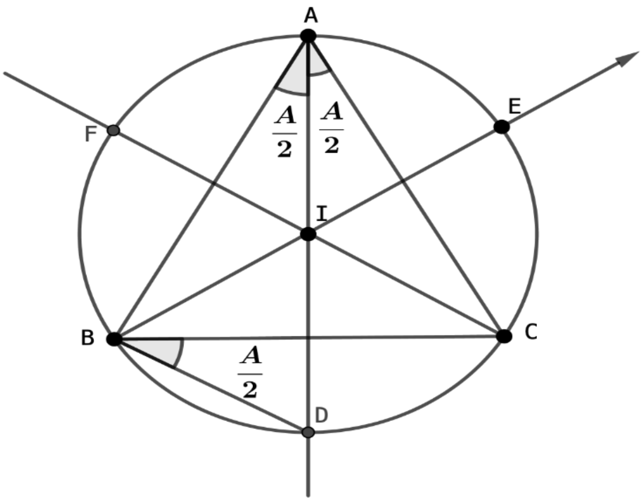
$$= (s^2 + r^2 - 8Rr)\cdot\frac{s^2+r^2-8Rr}{r^2} = \frac{(s^2+r^2-8Rr)^2}{r^2} \quad (6)$$

From (5) and (6), it follows that:

$$(ID+IE+IF)^2 \le \frac{(s^2+r^2-8Rr)^2}{4r^2}$$

$$ID+IE+IF \le \frac{s^2+r^2-8Rr}{2r} \overset{Gerretsen}{\le} \frac{4R^2-4Rr+4r^2}{2r} = \frac{2(R^2-Rr+r^2)}{r}$$

Solution 2 by Tapas Das – India



$\angle DBC = \angle DAC = \frac{A}{2}$ (angle on same one DC), $\angle ACB = \angle ADB = C$

(angle on same one AB). From ΔABD we have

$$\frac{C}{\sin C} = \frac{BD}{\sin\frac{A}{2}} = 2R \Rightarrow BD = 2R\sin\frac{A}{2}$$

Now from ΔBID,

$$\angle IBD = \frac{B}{2}+\frac{A}{2} = \frac{\pi}{2}-\frac{C}{2} \quad (\because A+B+C=\pi), \angle BDI = C$$

$$\therefore \angle BID = \pi - \left(\frac{\pi}{2}-\frac{C}{2}+C\right) = \frac{\pi}{2}-\frac{C}{2}$$

$$\therefore \angle BID = \angle DBI \quad \therefore BD = ID$$

$$\therefore ID = 2R\sin\frac{A}{2} \quad \text{(analog)}$$

$$\therefore ID+IE+IF = 2R\sin\frac{A}{2}+2R\sin\frac{B}{2}+2R\sin\frac{C}{2}$$

$$= 2R\left(\sum\sin\frac{A}{2}\right) \overset{Jensen's}{\le} 2R\,3\sin\left(\frac{A+B+C}{6}\right) = 2R\cdot 3\sin\frac{\pi}{6} = 3R$$

$[\sin\frac{x}{2}$ is concave in $x \in (0,\pi)]$

We need to show

$$3R \le \frac{2(R^2-Rr+r^2)}{r} \Rightarrow 2R^2-5Rr+2r^2 \ge 0 \Rightarrow (R-2r)(2R-r) \ge 0 \text{ True (Euler)}$$

Solution 3 by Marin Chirciu – Romania

Using the power of point I towards circumcircle we obtain

$IA \cdot ID = R^2 - OI^2$ and as $IA = \frac{r}{\sin\frac{A}{2}}, OI^2 = R^2 - 2R$ it follows $\frac{r}{\sin\frac{A}{2}} \cdot ID = 2Rr \Leftrightarrow$

$$\Leftrightarrow ID = 2R\sin\frac{A}{2}$$

We obtain:

$$LHS = \sum ID = \sum 2R\sin\frac{A}{2} = 2R\sum\sin\frac{A}{2} \le 2R\cdot\frac{3}{2} = 3R \overset{(1)}{\le} \frac{2(R^2-Rr+r^2)}{r}$$
$$= RHS,$$

where (1) $\Leftrightarrow 3R \le \frac{2(R^2-Rr+r^2)}{r} \Leftrightarrow 2R^2-5Rr+2r^2 \ge 0 \Leftrightarrow (R-2r)(2R-r) \ge 0,$

see $R \ge 2r$, (Euler).

Equality holds if and only if the triangle is equilateral.

JP.535 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{r_a^4 + r_b^2 r_c^2}{r_b^2 + r_c^2} \ge s^2$$

Proposed by Marian Ursărescu – Romania

Solution 1 by proposer

Lemma. If $x, y, z > 0$ then:

$$\sum_{cyc}\frac{x^2+yz}{y+z} \ge x+y+z$$

Proof of Lemma. We have:

$$\sum_{cyc}\frac{x^2+yz}{y+z} \ge x+y+z \Leftrightarrow \sum_{cyc}\frac{x^2+xy+yz+zx}{y+z} \ge 2(x+y+z)$$

$$\sum_{cyc}\frac{(x+y)(x+z)}{y+z} \ge (x+y)+(y+z)+(z+x)$$

which is true, because for $x+y=m, x+z=n, y+z=p$, we have:

$$\sum_{cyc}\frac{mn}{p}\geq m+n+p \Leftrightarrow (mn)^2+(np)^2+(pm)^2\geq mnp(m+n+p)$$

Hence,

$$\sum_{cyc}\frac{r_a^4+r_b^2r_c^2}{r_b^2+r_c^2}\geq r_a^2+r_b^2+r_c^2=(4R+r)^2-2s^2 \quad (1)$$

But: $(4R+r)^2\geq 3s^2$; (Doucet); (2). From (1) and (2), it follows that

$$\sum_{cyc}\frac{r_a^4+r_b^2r_c^2}{r_b^2+r_c^2}\geq s^2$$

Solution 2 by Marin Chirciu – Romania

$$LHS=\sum\frac{r_a^4+r_b^2r_c^2}{r_b^2+r_c^2}\overset{CS}{\geq}\frac{(\sum r_a^2)^2+(\sum r_br_c)^2}{\sum(r_b^2+r_c^2)}=\frac{(\sum r_a^2)^2+(s^2)^2}{2\sum r_a^2}\overset{(1)}{\geq}s^2=RHS$$

where (1) $\Leftrightarrow \frac{(\sum r_a^2)^2+s^4}{2\sum r_a^2}\geq s^2\Leftrightarrow(\sum r_a^2)^2+s^4\geq 2s^2\sum r_a^2\Leftrightarrow(\sum r_a^2-s^2)^2\geq 0$, with equality for $\sum r_a^2=s^2$.

Equality holds if and only if the triangle is equilateral.

JP.536 InΔABC the following relationship holds:

$$\frac{2R}{r}\geq\frac{(4R+r)^2}{s^2}+1$$

Proposed by Alex Szoros – Romania

Solution 1 by proposer

We will demonstrate beforehand that:

$$(\sum a)\left(\sum\frac{1}{a}\right)\geq\frac{2\sum a^2}{\sum ab}+7 \ (\forall)a,b,c>0 \quad (1)$$

The inequality is equivalent to

$$\frac{(\sum a)(\sum ab)}{abc}\geq\frac{2(\sum a^2)+7\sum ab}{\sum ab}\Leftrightarrow(\sum a)(\sum ab)^2\geq 2abc(\sum a^2)+7abc\sum ab \quad (2)$$

On the other hand

$$(\sum a)(\sum ab)^2\geq 3abc(\sum a)^2$$

It suffices to show that:

$$3abc\left(\sum a\right)^2 \ge 2abc\left(\sum a^2\right) + 7abc\sum ab \quad \Leftrightarrow$$

$$3abc\sum a^2 + 6abc\sum ab \ge 2abc\left(\sum a^2\right) + 7abc\sum ab \quad \Leftrightarrow abc\sum a^2 \ge abc\sum ab \quad \Leftrightarrow$$

$$\sum a^2 \ge \sum ab$$

and inequality (1) is proved.

From inequality (1) it follows that:

$$\left(\sum r_a\right)\left(\sum \frac{1}{r_a}\right) \ge \frac{2\sum r_a^2}{\sum r_a r_b} + 7 \quad \textbf{(3)}$$

$$\textbf{(3)} \Leftrightarrow \left(\sum r_a\right)\left(\sum \frac{1}{r_a}\right) \ge \frac{2\left(\sum r_a\right)^2}{\sum r_a r_b} + 3$$

Using identities

$$\sum r_a = 4R + r, \ \sum \frac{1}{r_a} = \frac{1}{r} \ \text{ and } \sum r_a r_b = s^2$$

relation (3) becomes

$$\frac{4R + r}{r} \ge \frac{2(4R + r)^2}{s^2} + 3 \quad \Leftrightarrow$$

$$\frac{4R}{r} \ge \frac{2(4R + r)^2}{s^2} + 2 \quad \Leftrightarrow \quad \frac{2R}{r} \ge \frac{(4R + r)^2}{s^2} + 1$$

Solution 2 by Marin Chirciu – Romania

Using Gerretsen's inequality $s^2 \ge 16Rr - 5r^2 \ge \frac{r(4R+r)^2}{R+r}$ **it suffices to prove that:**

$$\frac{2R}{r} \ge \frac{(4R+r)^2}{\frac{r(4R+r)^2}{R+r}} + 1 \Leftrightarrow \frac{2R}{r} \ge \frac{R+r}{r} + 1 \Leftrightarrow R \ge 2r, \text{ (Euler)}$$

Equality holds if and only if the triangle is equilateral.

Solution 3 by Tapas Das – India

$$\frac{2R}{r} \ge \frac{(4R+r)^2}{s^2} + 1 \text{ or } \frac{2R}{r} \cdot s^2 \ge (4R + r)^2 + s^2$$

$$\frac{2R}{r}(16Rr - sr^2) \overset{Gerretsen}{\ge} (4R + r)^2 + 4R^2 + 4Rr + 3r^2$$

$$\Rightarrow 32R^2 - 10Rr \ge 20R^2 + 12Rr + 4r^2 \Rightarrow 12R^2 - 22Rr - 4r^2 \ge 0$$

$$\text{or } 6R^2 - 11Rr - 2r^2 \ge 0 \ \text{ or } 6R^2 - 12Rr + Rr - 2r^2 \ge 0$$

$$\text{or } 6R(R - 2r) + r(R - 2r) \ge 0. \text{True (Euler)}$$

JP.537 Find the angles of a triangle ABC if

$$\frac{\sin A+2\sin B}{\sqrt{\sin^2 B+\sin^2 C+2\cos A\sin B\sin C}}+1=\frac{3\sqrt{3}}{2\sin C}$$

Proposed by Cristian Miu – Romania

Solution 1 by proposer

$h_a=2R\sin B\sin C\,,h_b=2R\sin A\sin C$ so we can write

$$\frac{h_b+2h_a}{\sqrt{b^2+c^2+2bc\cos A}}+\sin C=\frac{3\sqrt{3}}{2}\leftrightarrow$$

$$\frac{h_b+2h_a}{2m_a}+\sin C=\frac{3\sqrt{3}}{2}\leftrightarrow$$

$\frac{F}{bm_a}+\frac{2F}{am_a}+\sin C=\frac{3\sqrt{3}}{2}$ where F is the area of ABC

Let $M\in BC$ such as $BM=MC$ then

$$\frac{F}{bm_a}=\sin\left(\widehat{MAC}\right),\frac{2F}{am_a}=\sin\left(\widehat{AMC}\right)$$

Now we can write

$\sin\left(\widehat{MAC}\right)+\sin\left(\widehat{AMC}\right)+\sin C=\frac{3\sqrt{3}}{2}$. From here we obtain that $C=\frac{\pi}{3}$ and $AM=\frac{BC}{2}$.

But if

$AM=MC$ then $A=\frac{\pi}{2}$.

So, $A=\frac{\pi}{2},B=\frac{\pi}{C},C=\frac{\pi}{3}$

Solution 2 by Marin Chirciu – Romania

In ΔABC non-obtuse we have $\cos A\geq 0$, with equality for $A=90^\circ$

$$\sin^2 B+\sin^2 C+2\cos A\sin B\sin C\geq\sin^2 B+\sin^2 C+2\cdot 0\cdot\sin B\sin C=$$
$$=\sin^2 B+\sin^2 C$$

For $A=90^\circ\Rightarrow B+C=90^\circ\Rightarrow\sin^2 B+\sin^2 C=1$

We obtain:

$$LHS=\frac{\sin A+2\sin B}{\sqrt{\sin^2 B+\sin^2 C+2\cos A\sin B\sin C}}+1\leq\frac{\sin A+2\sin B}{\sqrt{1+2\cdot 0}}+1=$$

$$=\frac{1+2\sin B}{1}+1=2\sin B+2$$

We have $LHS\le \sin B+2$

From $RHS=\frac{3\sqrt{3}}{2\sin C}=LHS\ge 2\sin B+2\Rightarrow\frac{3\sqrt{3}}{2\sin C}\ge 2\sin B+2$, with equality for

$\sin B=\frac{1}{2}$ and $\sin C=\frac{\sqrt{3}}{2}\Rightarrow B=30^\circ$ and $C=60^\circ$

We deduce that $A=90^\circ, B=30^\circ$ and $c=60^\circ$.

JP.538 In ΔABC the following relationship holds:

$$\frac{3}{2R}\le\sum\frac{\cos^2\frac{A}{2}}{h_a}\le\frac{3}{4r}$$

Proposed by Alex Szoros – Romania

Solution 1 by proposer

Using the formulas

$\cos A=\frac{b^2+c^2-a^2}{2bc}$; $h_a=\frac{2F}{a}$; $R=\frac{abc}{4F}$ and $16F^2=2\sum a^2b^2-\sum a^4$

we have:

$$\sum\frac{\cos A}{h_a}=\sum\frac{a}{2F}\left(\frac{b^2+c^2-a^2}{2bc}\right)=\frac{1}{4Fabc}\sum a^2\left(b^2+c^2-a^2\right)=\frac{2\sum a^2b^2-\sum a^4}{16RF^2}=$$

$$=\frac{16F^2}{16RF^2}=\frac{1}{R}\quad(1)$$

Using the identity

$$\sum\frac{1}{h_a}=\frac{1}{r}\quad(2)$$

we can write that

$$\sum\frac{\cos A}{h_a}+\sum\frac{1}{h_a}=\frac{1}{R}+\frac{1}{r}\Rightarrow\sum\frac{1+\cos A}{h_a}=\frac{1}{R}+\frac{1}{r}\Rightarrow 2\sum\frac{\cos^2\frac{A}{2}}{h_a}=\frac{1}{R}+\frac{1}{r}\quad(3)$$

On the other hand from Euler's inequality $R\ge 2r$ we deduce

$$\frac{3}{R}\le\frac{1}{R}+\frac{1}{r}\le\frac{3}{2r}\Rightarrow\frac{3}{2R}\le\sum\frac{\cos^2\frac{A}{2}}{h_a}\le\frac{3}{4r}$$

Solution 2 by Marin Chirciu – Romania

Lemma.
In ΔABC:

$$\sum \frac{\cos^2\frac{A}{2}}{h_a} = \frac{R+r}{2Rr}$$

Proof.

$$\sum \frac{\cos^2\frac{A}{2}}{h_a} = \sum \frac{\frac{s(s-a)}{bc}}{\frac{2S}{a}} = \frac{s}{2S}\sum \frac{a(s-a)}{bc} = \frac{s}{2pr}\cdot\frac{R+r}{R} = \frac{R+r}{2Rr}$$

Let's get back to the main problem.Using the Lemma we obtain:
RHS.

$$\sum \frac{\cos^2\frac{A}{2}}{h_a} = \frac{R+r}{2Rr} \overset{Euler}{\leq} \frac{3}{4r}$$

Equality holds if and only if the triangle is equilateral.
LHS.

$$\sum \frac{\cos^2\frac{A}{2}}{h_a} = \frac{R+r}{2Rr} \overset{Euler}{\geq} \frac{3}{2R}$$

Equality holds if and only if the triangle is equilateral.

Solution 3 by Tapas Das – India

$$\text{WLOG } a \geq b \geq c, \cos^2\frac{A}{2} \leq \cos^2\frac{B}{2} \leq \cos^2\frac{C}{2}$$

$$h_a \leq h_b \leq h_c \Rightarrow \frac{1}{h_a} \geq \frac{1}{h_b} \geq \frac{1}{h_c}$$

$$\therefore \sum \frac{\cos^2\frac{A}{2}}{h_a} \overset{Chebyshev}{\leq} \frac{1}{3}\cdot\sum \cos^2\frac{A}{2}\sum\frac{1}{h_a} = \frac{1}{3}\left(2+\frac{r}{2R}\right)\cdot\frac{1}{r} \overset{Euler}{\leq} \frac{1}{3}\left(2+\frac{1}{4}\right)\cdot\frac{1}{r} = \frac{3}{4r}$$

$$\sum \frac{\cos^2\frac{A}{2}}{h_a} = \frac{1}{2F}\sum a\cos^2\frac{A}{2} \overset{AM-GM}{\geq} \frac{1}{2F}\cdot 3\left[(abc)\left(\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\right)\right]^{\frac{1}{3}} =$$

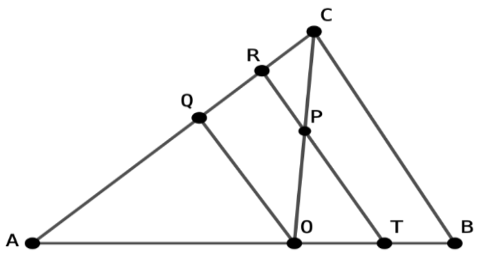
$$= \frac{1}{2F}\cdot 3\left[4Rrs\cdot\frac{s^2}{16R^2}\right]^{\frac{1}{3}} = \frac{1}{2F}\cdot 3\left[\frac{r^3s^3}{4R^2\cdot r^2}\right]^{\frac{1}{3}} \overset{Euler}{\geq} \frac{1}{2F}\cdot 3\left[\frac{r^3s^3}{4R^2\left(\frac{R}{2}\right)^2}\right]^{\frac{1}{3}} =$$

$$= \frac{1}{2F}\cdot 3\cdot\frac{rs}{R} = \frac{3}{2R}$$

JP.539 In $\Delta ABC, O \in (AB), OQ \parallel BC$, where $Q \in (AC). P \in (OC)$ such that $RP \parallel BC$, where $R \in (AC)$ and $T \in (AB)$. If the lengths of the segment RT is the geometric mean of the lengths of the segments OQ and BC, then $OP < \frac{OC}{2}$.

Proposed by Gheorghe Molea – Romania

Solution by proposer:



$$\text{From } OQ \parallel BC \Rightarrow \Delta AOQ \sim \Delta ABC \Rightarrow \frac{AO}{AB} = \frac{QO}{BC} \Rightarrow OQ = \frac{AO \cdot BC}{AB}$$

$$\text{From } RP \parallel BC \Rightarrow \Delta CRP \sim \Delta CQO \Rightarrow \frac{RP}{QO} = \frac{CP}{CO} \Rightarrow RP = \frac{CP \cdot QO}{CO}$$

$$\text{From } PT \parallel BC \Rightarrow \Delta OPT \sim \Delta OCB \Rightarrow \frac{OP}{OC} = \frac{PT}{BC} \Rightarrow PT = \frac{OP \cdot BC}{OC}$$

$$RT = RP + PT = \frac{CP \cdot QO + OP \cdot BC}{CO}$$

From $RT^2 = OQ \cdot BC$ (from hypotenuse) $\Rightarrow (CP \cdot QO + OP \cdot BC)^2 = OQ \cdot BC \cdot CO^2$

$$CP \cdot QO + OP \cdot BC = CO \cdot BC \cdot \sqrt{\frac{AO}{AB}},$$

$$CP \cdot \frac{AO \cdot BC}{AB} + OP \cdot BC = CO \cdot BC \cdot \sqrt{\frac{AO}{AB}} |: BC$$

$$CP \cdot AO + OP \cdot AB = CO \cdot \sqrt{AB \cdot AO}, \qquad CP \cdot AO + OP \cdot (AO + OB) = CO\sqrt{AB \cdot AO}$$

$$AO(CP + OP) + OP \cdot OB = CO \cdot \sqrt{AB \cdot AO}$$

$$AO \cdot CO + OP \cdot OB = CO \cdot \sqrt{AB \cdot AO}$$

$$\text{But } \sqrt{AB \cdot AO} < \frac{AB + AO}{2} \Rightarrow AO \cdot CO + OP \cdot OB < \frac{CO(AB + AO)}{2}$$

$$2AO \cdot CO + 2OP \cdot OB < CO \cdot AB + CO \cdot AO$$

$$AO\cdot CO+2OP\cdot OB<CO\cdot AB, 2OP\cdot OB<CO(AB-AO), 2OP\cdot OB<CO\cdot OB|:OB$$

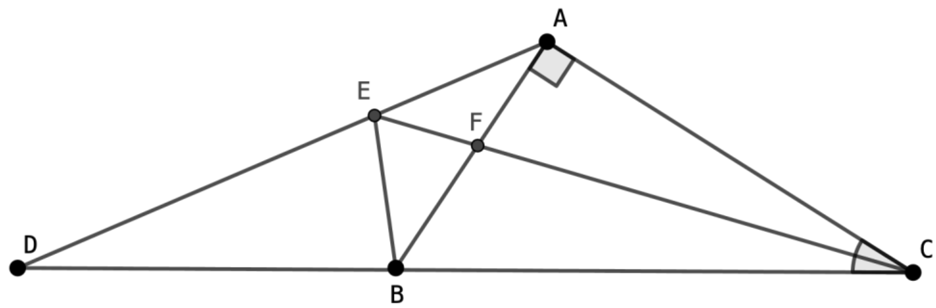
$$2OP<CO\Rightarrow OP<\frac{OC}{2}$$

JP.540 Let be ΔACD with $m(\widehat{CAD})>90^\circ, B\in(CD)$, such that $m(\widehat{BAC})=90^\circ$ and $AC>AB$. The bisector $\widehat{ACD}$ intersects AD in E. If $(BE$ is the bisector $\widehat{ABD}$, prove that:

$$\frac{1}{AD}=\frac{\sqrt{2}}{2}\left(\frac{1}{AB}-\frac{1}{AC}\right)$$

Proposed by Gheorghe Molea – Romania

Solution 1 by proposer



Let be $AB\cap CE=\{F\}$

$$(CF=\text{the bisector } \widehat{ACB}\Rightarrow\frac{AF}{FB}=\frac{CA}{CB}$$

$$(BE=\text{the bisector } \widehat{ABD}\Rightarrow\frac{DE}{EA}=\frac{BD}{AB}$$

$$(CE=\text{the bisector } \widehat{DCA}\Rightarrow\frac{CD}{CA}=\frac{DE}{EA}$$

$$\Rightarrow\frac{CD}{CA}=\frac{BD}{AB}\Leftrightarrow\frac{DB+BC}{CA}=\frac{BD}{AB}\Leftrightarrow\frac{DB}{CA}+\frac{BC}{CA}=\frac{BD}{AB}$$

$$\frac{BD}{AB}-\frac{DB}{AC}=\frac{BC}{CA}\Leftrightarrow BD\left(\frac{1}{AB}-\frac{1}{AC}\right)=\frac{FB}{FA}\Leftrightarrow$$

$$BD\left(\frac{1}{AB}-\frac{1}{AC}\right)=\frac{CB}{CA}\Rightarrow BD=\frac{CB}{CA}\cdot\frac{AB\cdot AC}{AC-AB}\Rightarrow BD=\frac{AB\cdot BC}{AC-AB}$$

$$DC=BD+BC=\frac{AB\cdot BC}{AC-AB}+BC=\frac{BC\cdot AC}{AC-AB}$$

Stewart's relationship in ΔADC:

$$AD^2\cdot BC+AC^2\cdot DB-AB^2\cdot DC=DB\cdot BC\cdot DC$$

$$AD^2\cdot BC+AC^2\cdot\frac{AB\cdot BC}{AC-AB}-AB^2\cdot\frac{BC\cdot AC}{AC-AB}=\frac{AB\cdot BC}{AC-AB}\cdot BC\cdot\frac{BC\cdot AC}{AC-AB}$$

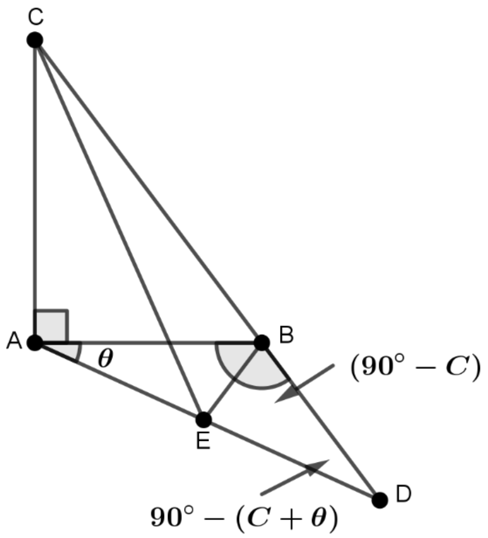
$$AD^2+AB\cdot AC\left(\frac{AC-AB}{AC-AB}\right)=\frac{AB\cdot AC\cdot BC^2}{(AC-AB)^2}$$

$$AD^2=AB\cdot AC\left(\frac{BC^2}{(AC-AB)^2}-1\right)=AB\cdot AC\left(\frac{BC^2-AC^2-AB^2+2BC\cdot AC}{(AC-AB)^2}\right)$$

$$=AB\cdot AC\cdot\frac{2AC\cdot AB}{(AC-AB)^2}=2\frac{AB^2\cdot AC^2}{(AC-AB)^2}$$

$$\Rightarrow AD=\frac{AB\cdot AC\sqrt{2}}{AC-AB}\Rightarrow\frac{1}{AD}=\frac{\sqrt{2}}{2}\left(\frac{AC\cdot AB}{AB\cdot AC}\right)\Rightarrow\frac{1}{AD}=\frac{\sqrt{2}}{2}\left(\frac{1}{AB}-\frac{1}{AC}\right)$$

Solution 2 by Debrata Nag-Kolkata-India



Clearly: $\frac{AE}{ED}=\frac{CA}{CD}=\frac{BA}{BD}$ **(by ∠ - bisector theorem)**

$$\therefore\frac{CA}{CD}=\frac{BA}{BD}=\frac{AC-AB}{BC}\Rightarrow BD=\frac{AB\cdot BC}{AC-AB}$$

Let $\angle BAD=\theta\Rightarrow\angle BDA=90^\circ-(C+\theta)$ **(1)**

$$\therefore\text{ from }\Delta ABD:\frac{AB}{AD}=\frac{\cos(C+\theta)}{\cos C}$$

$$\therefore\frac{AB}{AD}=\cos\theta-\tan C\sin\theta=\left(\cos\theta-\frac{AB}{AC}\sin\theta\right)\quad(2)$$

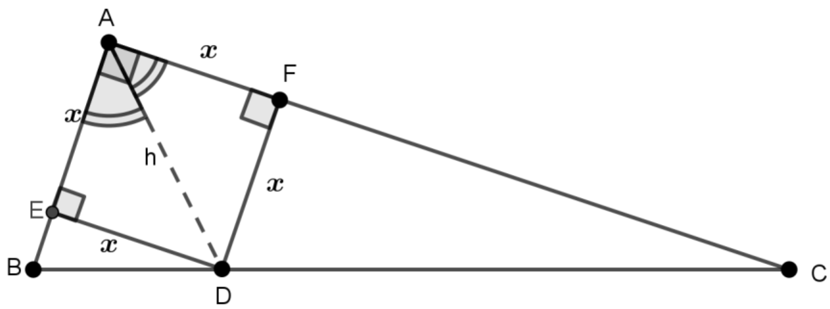
$$\text{Again, }\frac{AB}{BD}=\frac{\cos(C+\theta)}{\sin\theta}=\left(\frac{\cos C}{\tan\theta}-\sin C\right)$$

$$\text{from (1): } \frac{AB}{BD} = \frac{AC-AB}{BC} = \frac{AC}{BC} - \frac{AB}{BC} = \cos C - \sin C$$

$$\Rightarrow \frac{\cos C}{\tan\theta} - \sin C = \cos C = \sin C \Rightarrow \tan\theta = 1 \Rightarrow \theta = 45^\circ$$

$$\therefore \text{from (2): } \frac{AB}{AD} = \frac{1}{\sqrt{2}}\left(\frac{AC-AB}{AC}\right) \Rightarrow \frac{1}{AD} = \frac{\sqrt{2}}{2}\left(\frac{1}{AB} - \frac{1}{AC}\right)$$

Solution 3 by Alin Popa-Romania



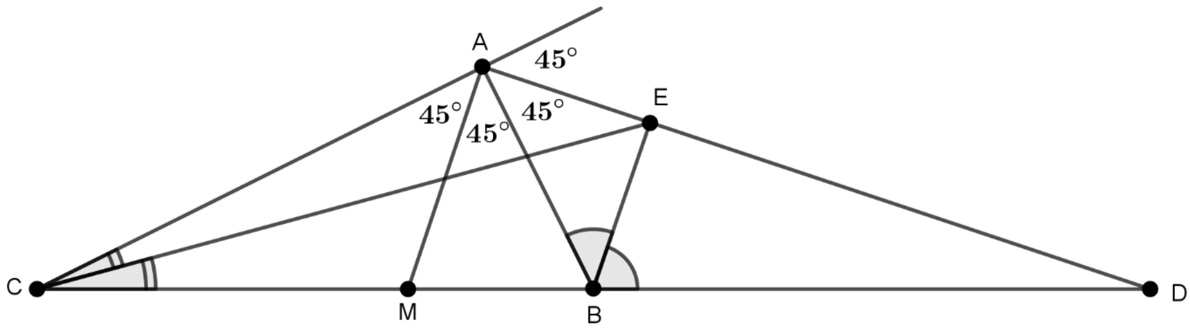
Lemma.

$$A = 90^\circ \Rightarrow h = \frac{h_c}{h+c}\sqrt{2} \Leftrightarrow \frac{1}{h_a} = \left(\frac{1}{h} + \frac{1}{c}\right)\frac{\sqrt{2}}{2}$$

Obs. $AEDF$ **rectangle and as** AD **bisector** $\Rightarrow AEDF$ **square**

Bisector Theorem $\Rightarrow BD = \frac{ac}{h+c}$; $\Delta BDE \sim \Delta BCA \Rightarrow \frac{DE}{AC} = \frac{BD}{BC}$

$$\Leftrightarrow \frac{x}{h} = \frac{ac}{(h+c)a} \Leftrightarrow x = \frac{h_c}{h+c} \Rightarrow h_a = \frac{h_c}{h+c}\sqrt{2}$$



$$\left.\begin{matrix} CE \text{ bisector } \measuredangle ACD \\ BE \text{ bisector } \measuredangle ABD \end{matrix}\right\} \Rightarrow AE \text{ exterior bisector } \measuredangle CAB$$

Let be AM **interior bisector** $\measuredangle CAB \Rightarrow AB$ **bisector** $\measuredangle MAD$

$$\left.\begin{array}{l}\frac{1}{AB}=\left(\frac{1}{AM}+\frac{1}{AD}\right)\frac{\sqrt{2}}{2}\\ \frac{1}{AM}=\left(\frac{1}{AB}+\frac{1}{AC}\right)\frac{\sqrt{2}}{2}\end{array}\right\}\overset{\ominus}{\Rightarrow}\frac{1}{AD}=\left(\frac{1}{AB}-\frac{1}{AC}\right)\frac{\sqrt{2}}{2}$$

PROBLEMS FOR SENIORS

SP.526 If $a,b,c,\lambda>0, a+b+c=\lambda$ then:

$$\sum\sqrt{\frac{bc}{a}+\lambda}\ge 2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)$$

Proposed by Marin Chirciu – Romania

Solution 1 by proposer

We obtain:

$$LHS=\sum\sqrt{\frac{bc}{a}+\lambda}=\sum\sqrt{\frac{bc}{a}+(a+b+c)}=\sum\sqrt{\frac{(a+b)(a+c)}{a}}\overset{(1)}{\ge}$$

$$\ge 2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)=RHS,$$

where (1) $\Leftrightarrow \sum\sqrt{\frac{(a+b)(a+c)}{a}}\ge 2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)\Leftrightarrow$

$\Leftrightarrow \sum\sqrt{bc(a+b)(a+c)}\ge 2\sqrt{abc}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)$, **which follows from:**

$$\sqrt{bc(a+b)(a+c)}\ge\sqrt{abc}\left(\sqrt{b}+\sqrt{c}\right)\Leftrightarrow\sqrt{(a+b)(a+c)}\ge\sqrt{a}\left(\sqrt{b}+\sqrt{c}\right)\Leftrightarrow$$

$$\Leftrightarrow (a+b)(a+c)\ge a\left(\sqrt{b}+\sqrt{c}\right)^2\Leftrightarrow\left(a-\sqrt{bc}\right)^2\ge 0.$$

Equality holds if and only if $a=b=c=\frac{\lambda}{3}$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By CBS inequality, we have

$$\sqrt{\frac{bc}{a}+\lambda}=\sqrt{\frac{bc}{a}+a+b+c}=\sqrt{\left(\frac{b}{a}+1\right)(a+c)}\ge\sqrt{b}+\sqrt{c}.$$

Similarly, we have

$$\sqrt{\frac{ca}{b}+\lambda} \ge \sqrt{c}+\sqrt{a} \text{ and } \sqrt{\frac{ab}{c}+\lambda} \ge \sqrt{a}+\sqrt{b}.$$

Adding these inequalities yields the desired result.

Equality holds iff $a = b = c = \frac{\lambda}{3}$.

SP.527 If $x, y, z \ge 0$ with $x + y + z = 1$ and $0 \le \lambda \le \frac{9}{4}$,

$$\text{then}: xy + yz + zx - \lambda xyz \le \frac{9-\lambda}{27}$$

Proposed by Marin Chirciu-Romania

Solution 1 by proposer

The inequality can be written equivalently. Denoting $yz = t$, the inequality can be written:

$$xy + t + zx - 2\lambda xt \le \frac{9-\lambda}{27} \Leftrightarrow 27t(\lambda x - 1) - 27x(y+z) + 9 - \lambda \ge 0 \Leftrightarrow$$

$$27t(\lambda x - 1) - 27x(1-x) + 9 - \lambda \ge 0 \Leftrightarrow 27t(\lambda x - 1) + 27x(x-1) + 9 - \lambda \ge 0.$$

We use Lemma.

If the function $f: \mathbb{R} \to \mathbb{R}, f(x) = ax + b$, meets the conditions

$f(\alpha) \ge 0$ and $f(\beta) \ge 0$, where $a, b, \alpha, \beta \in \mathbb{R}$, then $f(t) \ge 0, \forall t \in [\alpha, \beta]$.

Let's get back to the main problem.

Denoting $yz = t$ we have $t = yz \overset{AGM}{\le} \left(\frac{y+z}{2}\right)^2 \overset{x+y+z=1}{=} \left(\frac{1-x}{2}\right)^2 = t_0$.

$f(t) = 27t(\lambda x - 1) + 27x(x-1) + 9 - \lambda \ge 0$, where $0 \le t \le t_0 = \frac{(1-x)^2}{4}$.

We have:

$f(0) = 27x(x-1) + 9 - \lambda = 27x^2 - 27x + 9 - \lambda \ge 0$, because $\Delta = 27(4\lambda - 9) \le 0$.

$$f(t_0) = f\left(\frac{(1-x)^2}{4}\right) = \frac{1}{4}(27\lambda x^3 + (81 - 54\lambda)x^2 + (27\lambda - 54)x + 1) =$$

$$= \frac{1}{4}(3x-1)^2(3\lambda x + 9 - 4\lambda) \ge 0$$

Using Lemma for $f(t) = 27t(\lambda x - 1) + 27x(x-1) + 9 - \lambda \ge 0$ and $\alpha = 0, \beta = \frac{(1-x)^2}{4}$,

we deduce the conclusion.

Equality holds if and only if $x=y=z=\frac{1}{3}$, for $\lambda<\frac{9}{4}$, and for $\lambda=\frac{9}{4}$ equality for

$$(x,y,z)\in\left\{\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right),\left(0,\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{2},0,\frac{1}{2}\right),\left(\frac{1}{2},\frac{1}{2},0\right)\right\}.$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{Write the given inequality as follows}$$
$$\lambda\left(\frac{1}{27}-xyz\right)\le\frac{1}{3}-(xy+yz+zx).$$
$$\text{By AM}-\text{GM inequality, we have }3\sqrt[3]{xyz}\le x+y+z=1,\text{then }xyz\le\frac{1}{27}.$$
$$\text{So it suffices to prove that}$$
$$\frac{9}{4}\left(\frac{1}{27}-xyz\right)\le\frac{1}{3}-(xy+yz+zx)$$
$$\Leftrightarrow 4(xy+yz+zx)\le 1+9xyz$$
$$\Leftrightarrow 4(xy+yz+zx)(x+y+z)\le(x+y+z)^3+9xyz$$
$$\Leftrightarrow xy(x+y)+yz(y+z)+zx(z+x)\le x^3+y^3+z^3+3xyz,$$
$$\text{which is Schur's inequality.}$$
$$\text{So the proof is complete. Equality holds iff }x=y=z=\frac{1}{3}.$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$xy+yz+zx-\lambda xyz-\frac{9-\lambda}{27}\overset{x+y+z=1}{=}$$
$$\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)-\lambda xyz-\left(\frac{9-\lambda}{27}\right)\left(\sum_{cyc}x\right)^3$$
$$=\lambda\left(\frac{\left(\sum_{cyc}x\right)^3}{27}-xyz\right)-\left(\sum_{cyc}x\right)\left(\frac{\left(\sum_{cyc}x\right)^2}{3}-\sum_{cyc}xy\right)$$
$$=\frac{\lambda}{27}\cdot\left(\left(\sum_{cyc}x\right)^3-27xyz\right)-\frac{\left(\sum_{cyc}x\right)}{3}\cdot\left(\left(\sum_{cyc}x\right)^2-3\sum_{cyc}xy\right)$$
$$\overset{0\le\lambda\le\frac{9}{4}}{\le}\frac{\frac{9}{4}}{27}\cdot\left(\left(\sum_{cyc}x\right)^3-27xyz\right)-\frac{\left(\sum_{cyc}x\right)}{3}\cdot\left(\left(\sum_{cyc}x\right)^2-3\sum_{cyc}xy\right)\overset{?}{\le}0$$
$$\Leftrightarrow 4\left(\sum_{cyc}x\right)\left(\left(\sum_{cyc}x\right)^2-3\sum_{cyc}xy\right)-\left(\sum_{cyc}x\right)^3+27xyz\overset{?}{\ge}0$$
$$\Leftrightarrow\sum_{cyc}x^3+3xyz\overset{?}{\ge}\sum_{cyc}x^2y+\sum_{cyc}xy^2\to\text{true via Schur}\therefore xy+yz+zx-\lambda xyz$$

$$\le \frac{9-\lambda}{27}\ \forall\, x,y,z\ge 0 \text{ with } x+y+z=1 \text{ and } 0\le\lambda\le\frac{9}{4},$$

$$''='' \text{ iff } \left(x=y=z=\frac{1}{3},\lambda=\frac{9}{4}\right) \text{ or } \left(x=0,y=z=\frac{1}{2},\lambda=\frac{9}{4}\right)$$

$$\text{or } \left(y=0,z=x=\frac{1}{2},\lambda=\frac{9}{4}\right) \text{ or } \left(z=0,x=y=\frac{1}{2},\lambda=\frac{9}{4}\right) (\text{QED})$$

SP.528 If $a,b,c\ge 1$ then:

$$\frac{1}{9}(a+b+c)+\frac{1}{3\sqrt{2}}\ge\frac{\sqrt[3]{ab-1}}{b+c+\sqrt{2}}+\frac{\sqrt[3]{bc-1}}{c+a+\sqrt{2}}+\frac{\sqrt[3]{ca-1}}{a+b+\sqrt{2}}$$

Proposed by Marin Chirciu – Romania

Solution 1 by proposer

$$RHS=\sum\frac{\sqrt[3]{ab-1}}{b+c+\sqrt{2}}\overset{AGM}{\le}\sum\frac{1}{3}\sqrt[3]{\frac{ab-1}{\sqrt{2}bc}}=\frac{1}{3}\sum\sqrt[3]{\frac{1}{\sqrt{2}}\left(a-\frac{1}{b}\right)\frac{1}{c}}\overset{AGM}{\le}$$

$$\overset{AGM}{\le}\frac{1}{3}\sum\frac{\frac{1}{\sqrt{2}}+\left(a-\frac{1}{b}\right)+\frac{1}{c}}{3}=\frac{a+b+c+\frac{3}{\sqrt{2}}}{9}=LHS$$

with equality for $b=c=\sqrt{2},\frac{1}{\sqrt{2}}=\left(a-\frac{1}{b}\right)=\frac{1}{c}$.

Equality holds if and only if $a=b=c=\sqrt{2}$.

Solution 2 by Amir Sofi-Kosovo

$$b+c+\sqrt{2}\overset{AM-GM}{\ge}3\sqrt[3]{\sqrt{2}bc}$$

$$\Rightarrow\frac{\sqrt[3]{ab-1}}{b+c+\sqrt{2}}\le\frac{\sqrt[3]{ab-1}}{3\sqrt[3]{\sqrt{2}bc}}=\frac{3}{9}\sqrt[3]{\left(a-\frac{1}{b}\right)\frac{1}{c}\frac{1}{\sqrt{2}}}\le\frac{a-\frac{1}{b}+\frac{1}{c}+\frac{1}{\sqrt{2}}}{9}$$

$$\frac{\sqrt[3]{bc-1}}{c+a+\sqrt{2}}\le\frac{\sqrt[3]{bc-1}}{3\sqrt[3]{\sqrt{2}ca}}=\frac{3}{9}\sqrt[3]{\left(b-\frac{1}{c}\right)\frac{1}{a}\frac{1}{\sqrt{2}}}\le\frac{b-\frac{1}{c}+\frac{1}{a}+\frac{1}{\sqrt{2}}}{9}$$

$$\frac{\sqrt[3]{ca-1}}{a+b+\sqrt{2}}\le\frac{\sqrt[3]{ca-1}}{3\sqrt[3]{\sqrt{2}ab}}=\frac{3}{9}\sqrt[3]{\left(c-\frac{1}{a}\right)\frac{1}{b}\frac{1}{\sqrt{2}}}\le\frac{c-\frac{1}{a}+\frac{1}{b}+\frac{1}{\sqrt{2}}}{9}$$

Add 3 Inequalities we get:

$$\frac{\sqrt[3]{ab-1}}{b+c+\sqrt{2}}+\frac{\sqrt[3]{bc-1}}{c+a+\sqrt{2}}+\frac{\sqrt[3]{ca-1}}{a+b+\sqrt{2}}\le\frac{1}{9}(a+b+c)+\frac{1}{3\sqrt{2}}$$

SP.529 Let ABC be a triangle with inradius r and circumradius R and let the interior points D,E,F be chosen on the sides BC,CA,AB respectively, so that AD,BE,CF are the bisectors of the triangle ABC.Let r_A,r_B,r_C be the inradii of the triangles AEF,BFD,CDE respectively. Prove that:

$$r_A^2+r_B^2+r_C^2\le\frac{3R^4}{64r^2}$$

Proposed by George Apostolopoulos –Greece

Solution 1 by proposer

Let $\alpha=BC,b=CA,c=AB$ be the lengths of the sides and let R_A,R_B,R_C be the circumradii of the triangles AEF,BFD,CDE respectively.

Then $FE=2R_A\cdot\sin A$. We'll prove that $FE\le\frac{2a+b+c}{8}$. By the law of cosines in triangle

$$AEF: EF^2=AE^2+AF^2-2AE\cdot AF\cdot\cos A=$$

$$=\left(\frac{bc}{a+c}\right)^2+\left(\frac{bc}{a+b}\right)^2-2\left(\frac{bc}{a+c}\right)\cdot\left(\frac{bc}{a+b}\right)\cdot\frac{b^2+c^2-a^2}{2bc}=$$

$$=\frac{b^2c^2}{(a+c)^2}+\frac{b^2c^2}{(a+b)^2}-\frac{bc[(b-c)^2+2bc-a^2]}{(a+b)(a+c)}=$$

$$=b^2c^2\left(\frac{1}{a+c}-\frac{1}{a+b}\right)^2-\frac{bc(b-c)^2}{(a+b)(a+c)}+\frac{a^2bc}{(a+b)(a+c)}=$$

$$=\frac{b^2c^2(b-c)^2}{(a+b)^2(a+c)^2}-\frac{bc(b-c)^2}{(a+b)(a+c)}+\frac{a^2bc}{(a+b)(a+c)}=$$

$$=\frac{a^2bc}{(a+b)(a+c)}-\frac{bc(b-c)^2[(a+b)(a+c)-bc]}{(a+b)^2(a+c)^2}\le\frac{a^2bc}{(a+b)(a+c)}\le$$

$$\le\frac{a^2bc}{2\sqrt{ab}\cdot2\sqrt{bc}}=\frac{a\sqrt{bc}}{4}$$

So $EF\le\frac{\sqrt{a\sqrt{bc}}}{2}=\frac{2\sqrt{a}\sqrt{\sqrt{bc}}}{4}\le\frac{a+\sqrt{bc}}{4}\le\frac{a+\frac{b+c}{2}}{4}=\frac{2a+b+c}{8}$. Namely

$EF\le\frac{2a+b+c}{8}$ and analogs, $FD\le\frac{a+2b+c}{8}$, $DE\le\frac{a+b+2c}{8}$.

Now, we have $2R_A \cdot \sin A \le \frac{2a+b+c}{8}$.

But $R_A \ge 2r_A$ (EULER), so $4r_A \cdot \frac{a}{2R} \le \frac{2a+b+c}{8} \Leftrightarrow r_A \le \frac{R}{16}\left(2+\frac{b}{a}+\frac{c}{a}\right)$. Similarly,

$r_B \le \frac{R}{16}\left(2+\frac{c}{b}+\frac{a}{b}\right)$, and $r_C \le \frac{R}{16}\left(2+\frac{a}{c}+\frac{b}{c}\right)$. Now, we have

$$r_A^2+r_B^2+r_C^2 \le \frac{R^2}{256}\left[\left(2+\frac{b}{a}+\frac{c}{a}\right)^2+\left(2+\frac{c}{b}+\frac{a}{b}\right)^2+\left(2+\frac{a}{c}+\frac{b}{c}\right)^2\right]=$$

$$=\frac{R^2}{256}\left[\begin{array}{c}4+4\left(\frac{b}{a}+\frac{c}{a}\right)+\frac{b^2}{a^2}+\frac{c^2}{a^2}+\frac{2bc}{a^2}+4+4\left(\frac{c}{b}+\frac{a}{b}\right)+\frac{c^2}{b^2}+\frac{a^2}{b^2}+\frac{2ca}{b^2}+\\ +4+4\left(\frac{a}{c}+\frac{b}{c}\right)+\frac{a^2}{c^2}+\frac{b^2}{c^2}+\frac{2ab}{c^2}\end{array}\right]=$$

$$=\frac{R^2}{256}\left[\begin{array}{c}12+4\left(\frac{a}{b}+\frac{b}{a}+\frac{b}{c}+\frac{c}{b}+\frac{c}{a}+\frac{a}{c}\right)+\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)+\left(\frac{b^2}{c^2}+\frac{c^2}{b^2}\right)+\left(\frac{c^2}{a^2}+\frac{a^2}{c^2}\right)+\\ +\left(\frac{2bc}{a^2}+\frac{2ca}{b^2}+\frac{2ab}{c^2}\right)\end{array}\right]\le$$

$$\frac{R^2}{256}\left[\begin{array}{c}12+4\left(\frac{a}{b}+\frac{b}{a}+\frac{b}{c}+\frac{c}{b}+\frac{c}{a}+\frac{a}{c}\right)+\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)+\left(\frac{b^2}{c^2}+\frac{c^2}{b^2}\right)+\left(\frac{c^2}{a^2}+\frac{a^2}{c^2}\right)+\\ +\frac{b^2+c^2}{a^2}+\frac{c^2+a^2}{b^2}+\frac{a^2+b^2}{c^2}\end{array}\right]=$$

$$\frac{R^2}{256}\left[\begin{array}{c}12+4\left(\frac{a}{b}+\frac{b}{a}+\frac{b}{c}+\frac{c}{b}+\frac{c}{a}+\frac{a}{c}\right)+2\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)+2\left(\frac{b^2}{c^2}+\frac{c^2}{b^2}\right)+\\ +2\left(\frac{c^2}{a^2}+\frac{a^2}{c^2}\right)\end{array}\right]$$

It is well-known that: $\frac{a}{b}+\frac{b}{a} \le \frac{R}{r}, \frac{b}{c}+\frac{c}{b} \le \frac{R}{r}$, and $\frac{c}{a}+\frac{a}{c} \le \frac{R}{r}$. So

$$\frac{a^2}{b^2}+\frac{b^2}{a^2} \le \frac{R^2}{r^2}-2, \frac{b^2}{c^2}+\frac{c^2}{b^2} \le \frac{R^2}{r^2}-2, \text{ and } \frac{c^2}{a^2}+\frac{a^2}{c^2} \le \frac{R^2}{r^2}-2$$

Namely:

$$r_A^2+r_B^2+r_C^2 \le \frac{R^2}{256}\cdot\left[12+4\cdot 3\frac{R}{r}+6\left(\frac{R^2}{r^2}-2\right)\right]=$$

$$\frac{R^2}{256}\cdot 6\left(\frac{2R}{r}+\frac{R^2}{r^2}\right)=\frac{3R^2}{128}\cdot\frac{R}{r}\left(2+\frac{R}{r}\right)=\frac{3R^2}{128}\cdot\frac{R}{r}\cdot\frac{2r+R}{r}\overset{EULER}{\le}\frac{3R^3}{128r^2}(R+R)=\frac{3R^4}{64r^2}$$

So $r_A^2+r_B^2+r_C^2 \le \frac{3R^4}{64r^2}$. Equality holds iff the triangle ABC is equilateral.

Solution 2 by Tapas Das – India

R_A, R_B, R_c are the circumradius of $\Delta AEF, \Delta BFD, \Delta CDE$ respectively

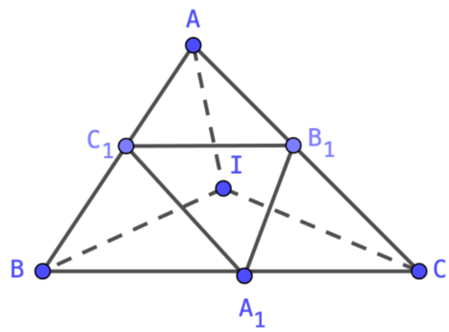
$$\therefore R_A \le \frac{B_1C_1}{2\sin A} \quad \therefore R_A^2 \le \frac{B_1C_1^2}{4\sin^2 A} \quad \therefore R_A^2 \le \frac{(a^2bc)^{\frac{1}{2}}}{16\sin^2 A} = \frac{R^2\sqrt{bc}}{4a}$$

$$\therefore r_A^2 + r_B^2 + r_C^2 \le \left(\frac{R_A}{2}\right)^2 + \left(\frac{R_B}{2}\right)^2 + \left(\frac{R_C}{2}\right)^2 = \frac{1}{4}[R_A^2 + R_B^2 + R_C^2]$$

$$\le \frac{1}{4}R^2\left[\frac{\sqrt{bc}}{4a} + \frac{\sqrt{ca}}{4b} + \frac{\sqrt{ab}}{4c}\right] = \frac{1}{16}R^2\left[\frac{\sqrt{bc}}{a} + \frac{\sqrt{ca}}{b} + \frac{\sqrt{ab}}{c}\right] \overset{CBS}{\le} \frac{1}{16}R^2\sqrt{\left(\sum bc\right)\cdot\sum\frac{1}{a^2}}$$

$$\le \frac{R^2}{16}\sqrt{\sum a^2 \cdot \sum \frac{1}{a^2}} \qquad \left[\because \sum bc \le \sum a^2\right]$$

$$\overset{Steining\ Leibniz}{\le} \frac{R^2}{16}\sqrt{\frac{9R^2}{4r^2}} = \frac{R^2}{16}\cdot\frac{3R}{2r} = \frac{3R^3}{32r} = \frac{3R^4}{32rR} \overset{Euler}{\le} \frac{3R^4}{64r^2}$$



$$AB_1 = \frac{bc}{c+a}, AC_1 = \frac{bc}{a+b}$$

$$B_1C_1^2 = AC_1^2 + AB_1^2 - 2AC_1AB_1\cos A$$

$$= \left(\frac{bc}{c+a}\right)^2 + \left(\frac{bc}{a+b}\right)^2 - 2\left(\frac{bc}{a+c}\right)\left(\frac{bc}{a+b}\right)\frac{b^2+c^2-a^2}{2bc}$$

$$= \frac{bc[bc(a+b)^2 + bc(c+a)^2 - (a+b)(a+c)(b^2+c^2-a^2)]}{(a+b)^2(a+c)^2}$$

$$= \frac{bc[a^2(a+b)(a+c) - (b^2+c^2)(a+b)(a+c) + bc(a+b)^2 + bc(c+a)^2]}{(a+b)^2(a+c)^2}$$

$$= \frac{bc[a^2(a+b)(a+c) - a(b-c)^2(a+b+c)]}{(a+b)^2(a+c)^2}$$

$$\leq \frac{bca^2(a+b)(a+c)}{(a+b)^2(a+c)^2} = \frac{a^2bc}{(a+b)(a+c)}$$

Note: $bc(a+b)^2 + bc(c+a)^2 - (b^2+c^2)(a+b)(a+c)$

$$= bc[(b-c)^2 + 2(a+b)(a+c)] - [(b-c)^2 + 2bc](a+b)(a+c)$$

$$= bc(b-c)^2 + 2bc(a+b)(a+c) - (b-c)^2(a+b)(a+c) - 2bc(a+b)(a+c)$$

$$= (b-c)^2[bc - (a+b)(a+c)]$$

$$= (b-c)^2[bc - a^2 - ac - ba - bc] = -a(b-c)^2(a+b+c)$$

$$\therefore B_1C_1^2 \leq \frac{a^2bc}{(a+b)(a+c)} \leq \frac{a^2bc}{4a\sqrt{bc}} \quad \text{(AM-GM)} = \frac{a\sqrt{bc}}{4} \therefore B_1C_1 \leq \frac{(a^2bc)^{\frac{1}{4}}}{2}$$

$$\therefore R_A = \frac{B_1C_1}{2\sin A}$$

Solution 3 by Marin Chirciu – Romania

Lemma.

In $\Delta ABC, AD, BE, CF$ – internal bisectors, r_A – inradii ΔAEF:

$$r_A^2 \leq \frac{R^2bc}{4(a+b)(a+c)}$$

Proof.

Let R_A, R_B, R_C – circumradii $\Delta AEF, \Delta BFD, \Delta CDE$.

$$R_A = \frac{EF}{2\sin A}$$

Lemma 1.

In $\Delta ABC, AD, BE, CF$ – internal bisectors

$$EF^2 \leq \frac{a^2bc}{(a+b)(a+c)}$$

Proof.

With bisector theorem we have $AD = \frac{bc}{a+c}, AF = \frac{bc}{a+b}$.

Using cosine theorem in ΔAEF we obtain:

$$EF^2 = AE^2 + AF^2 - 2AE \cdot AF \cdot \cos A =$$

$$= \left(\frac{bc}{a+c}\right)^2 + \left(\frac{bc}{a+b}\right)^2 - 2\frac{bc}{a+c} \cdot \frac{bc}{a+b} \cdot \frac{b^2+c^2-a^2}{2bc} =$$

$$=\frac{b^2c^2}{(a+c)^2}+\frac{b^2c^2}{(a+b)^2}-\frac{bc(b^2+c^2-a^2)}{(a+b)(a+c)}=$$

$$=\frac{b^2c^2}{(a+c)^2}+\frac{b^2c^2}{(a+b)^2}-\frac{bc[2bc+(b-c)^2-a^2]}{(a+b)(a+c)}=$$

$$=\frac{b^2c^2}{(a+c)^2}+\frac{b^2c^2}{(a+b)^2}-\frac{2b^2c^2}{(a+b)(a+c)}-\frac{bc(b-c)^2}{(a+b)(a+c)}+\frac{a^2bc}{(a+b)(a+c)}=$$

$$=\underbrace{\frac{b^2c^2}{(a+c)^2}+\frac{b^2c^2}{(a+b)^2}-\frac{2b^2c^2}{(a+b)(a+c)}}-\frac{bc(b-c)^2}{(a+b)(a+c)}+\frac{a^2bc}{(a+b)(a+c)}=$$

$$=b^2c^2\left(\frac{1}{a+c}-\frac{1}{a+b}\right)^2-\frac{bc(b-c)^2}{(a+b)(a+c)}+\frac{a^2bc}{(a+b)(a+c)}=$$

$$=\frac{b^2c^2(b-c)^2}{(a+b)^2(a+c)^2}-\frac{bc(b-c)^2}{(a+b)(a+c)}+\frac{a^2bc}{(a+b)(a+c)}=$$

$$=\frac{bc(b-c)^2[bc-(a+b)(a+c)]}{(a+b)^2(a+c)^2}+\frac{a^2bc}{(a+b)(a+c)}=$$

$$=\frac{bc(b-c)^2(-a^2-ab-ac)}{(a+b)^2(a+c)^2}+\frac{a^2bc}{(a+b)(a+c)}=$$

$$=\frac{-abc(b-c)^2(a+b+c)}{(a+b)^2(a+c)^2}+\frac{a^2bc}{(a+b)(a+c)}\le\frac{a^2bc}{(a+b)(a+c)}$$

Equality holds if and only if the triangle is equilateral.

We have $r_A\le\frac{R_A}{2}$ **and** $R_A=\frac{EF}{2\sin A}\Rightarrow r_A=\frac{EF}{4\sin A}\Rightarrow$

$$r_a^2=\frac{EF^2}{16\sin^2 A}=\frac{EF^2}{16\frac{a^2}{4R^2}}=\frac{R^2EF^2}{4a^2}\overset{Lemma\ 1}{\le}\frac{R^2\cdot\frac{a^2bc}{(a+b)(a+c)}}{4a^2}=\frac{R^2bc}{4(a+b)(a+c)}$$

Let's get back to the main problem.

Using Lemma $r_A^2\le\frac{R^2bc}{4(a+b)(a+c)}$ **we obtain:**

$$\sum r_A^2\overset{Lemma}{\le}\sum\frac{R^2bc}{4(a+b)(a+c)}=R^2\frac{\sum bc(b+c)}{4\prod(b+c)}=$$

$$=\frac{R^2}{4}\cdot\frac{2p(p^2-r^2-2Rr)}{2p(p^2+r^2+2Rr)}=\frac{R^2(p^2+r^2-2Rr)}{4(p^2+r^2+2Rr)}$$

We prove that:

$$\frac{R^2(p^2+r^2-2Rr)}{4(p^2+r^2+2Rr)} \le \frac{3R^4}{64r^2} \Leftrightarrow 16r^2(p^2+r^2-2Rr) \le 3R^2(p^2+r^2+2Rr) \Leftrightarrow$$

$$\Leftrightarrow p^2(3R^2-16r^2)+r(6R^3+3R^2r+32Rr^2-16r^3) \ge 0$$

We distinguish the cases:

Case 1. If $(3R^2-16r^2) \ge 0$ the inequality is obvious.

Case 2. If $(3R^2-16r^2) < 0$ the inequality can be written:

$r(6R^3+3R^2r+32Rr^2-16r^3) \ge p^2(16r^2-3R^2)$, which follows from Gerretsen's inequality: $p^2 \le 4R^2+4Rr+3r^2$.

It remains to prove that:

$$r(6R^3+3R^2r+32Rr^2-16r^3) \ge (4R^2+4Rr+3r^2)(16r^2-3R^2) \Leftrightarrow$$

$$\Leftrightarrow 6R^4+9R^3r-26R^2r^2-16Rr^3-32r^4 \ge 0 \Leftrightarrow$$

$\Leftrightarrow (R-2r)(6R^3+21Rr^2+16Rr^2+16r^3) \ge 0$, see $R \ge 2r$, (Euler).

Equality holds if and only if the triangle is equilateral.

SP.530 Let a,b,c be the lengths of the sides of a triangle with inradius circumradius R. Prove that:

$$\frac{3^{1-\frac{1}{2n}}}{\sqrt[n]{(m+1)R}} \le \frac{1}{\sqrt[n]{m\cdot a+b}}+\frac{1}{\sqrt[n]{m\cdot b+c}}+\frac{1}{\sqrt[n]{m\cdot c+a}} \le \frac{3^{1-\frac{1}{2n}}}{\sqrt[n]{(m+1)\cdot 2r}}$$

for all integers $m \ge 0$ and $n \ge 1$.

Proposed by George Apostolopoulos – Greece

Solution 1 by proposer

For the right inequality, first will prove that

$\frac{1}{m\cdot a+b} \le \frac{1}{(m+1)^2}\left(\frac{m}{a}+\frac{1}{b}\right)$. We have

$$\frac{1}{(m+1)^2}\left(\frac{m}{a}+\frac{1}{b}\right)-\frac{1}{m\cdot a+b} = \frac{(a+mb)(ma+b)-(m+1)^2ab}{(m+1)^2ab(ma+b)} =$$

$\frac{m(a-b)^2}{(m+1)^2ab(ma+b)} \ge 0$. So

$\frac{1}{\sqrt[n]{ma+b}} \le \frac{1}{\sqrt[n]{(m+1)^2}} \sqrt[n]{\frac{m}{a}+\frac{1}{b}}$. Similarly

$$\frac{1}{\sqrt[n]{mb+c}} \le \frac{1}{\sqrt[n]{(m+1)^2}} \sqrt[n]{\frac{m}{b}+\frac{1}{c}}, \frac{1}{\sqrt[n]{mc+a}} \le \frac{1}{\sqrt[n]{(m+1)^2}} \sqrt[n]{\frac{m}{c}+\frac{1}{a}}$$

Adding up these inequalities, we have

$$\frac{1}{\sqrt[n]{ma+b}}+\frac{1}{\sqrt[n]{mb+c}}+\frac{1}{\sqrt[n]{mc+a}} \le \frac{1}{\sqrt[n]{(m+1)^2}}\left(\sqrt[n]{\frac{m}{a}+\frac{1}{b}}+\sqrt[n]{\frac{m}{b}+\frac{1}{c}}+\sqrt[n]{\frac{m}{c}+\frac{1}{a}}\right)$$

We know that $\left(\frac{x+y+z}{3}\right)^n \le \frac{x^n+y^n+z^n}{3}, (x,y,z>0)$. So for

$x=\sqrt[n]{\frac{m}{a}+\frac{1}{b}}, y=\sqrt[n]{\frac{m}{b}+\frac{1}{c}}, z=\sqrt[n]{\frac{m}{c}+\frac{1}{a}}$, we get

$$\left(\frac{\sqrt[n]{\frac{m}{a}+\frac{1}{b}}+\sqrt[n]{\frac{m}{b}+\frac{1}{c}}+\sqrt[n]{\frac{m}{c}+\frac{1}{a}}}{3}\right)^n \le \frac{(m+1)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}{3} \text{ so}$$

$$\sqrt[n]{\frac{m}{a}+\frac{1}{b}}+\sqrt[n]{\frac{m}{b}+\frac{1}{c}}+\sqrt[n]{\frac{m}{c}+\frac{1}{a}} \le \frac{3}{\sqrt[n]{3}}\sqrt[n]{m+1}\cdot\sqrt[n]{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$$

We know that $\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2 \le 3\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)$ and

$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \le \frac{1}{4r^2}$ (easy proof), so $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \le \frac{\sqrt{3}}{2r}$. Now we have

$$\sqrt[n]{\frac{m}{a}+\frac{1}{b}}+\sqrt[n]{\frac{m}{b}+\frac{1}{c}}+\sqrt[n]{\frac{m}{c}+\frac{1}{a}} \le \frac{1}{\sqrt[n]{(m+1)^2}}\cdot\frac{3}{\sqrt[n]{3}}\sqrt[n]{m+1}\cdot\frac{\sqrt[n]{\sqrt{3}}}{\sqrt[n]{2r}} = \frac{3^{1-\frac{1}{2n}}}{\sqrt[n]{(m+1)\cdot 2r}}$$

For the left inequality, we use the AM-GM inequality, we have

$$\frac{1}{\sqrt[n]{ma+b}}+\frac{1}{\sqrt[n]{mb+c}}+\frac{1}{\sqrt[n]{mc+a}} \ge \frac{3}{\sqrt[3]{\sqrt[n]{(ma+b)(mb+c)(mc+a)}}} =$$

$$\frac{3}{\sqrt[n]{\sqrt[3]{(ma+b)(mb+c)(mc+a)}}} \ge \frac{3}{\sqrt[n]{\frac{(ma+b)+(mb+c)+(mc+a)}{3}}} =$$

$$\frac{3\sqrt[n]{3}}{\sqrt[n]{(m+1)(a+b+c)}} \ge \frac{3\sqrt[n]{3}}{\sqrt[n]{m+1}\cdot\sqrt[n]{3\sqrt{3}R}} = \frac{3^{1-\frac{1}{2n}}}{\sqrt[n]{(m+1)R}}$$

$$\left(a+b+c\le 3\sqrt{3}R\right)$$

Equality holds if and only if the triangle is equilateral.

Solution 2 by Tapas Das – India

$$\frac{1}{\sqrt[n]{ma+b}}+\frac{1}{\sqrt[n]{mb+c}}+\frac{1}{\sqrt[n]{mc+a}}=\frac{(1)^{\frac{1}{n}+1}}{\sqrt[n]{ma+b}}+\frac{(1)^{\frac{1}{n}+1}}{\sqrt[n]{mb+c}}+\frac{(1)^{\frac{1}{n}+1}}{\sqrt[n]{mc+a}}\ge$$

$$\overset{Radon}{\ge}\frac{(1+1+1)^{\frac{1}{n}+1}}{\left[m(\sum a)+\sum a\right]^{\frac{1}{n}}}=\frac{3^{\frac{1}{n}+1}}{(\sum a)^{\frac{1}{n}}\cdot(m+1)^{\frac{1}{n}}}=\frac{3^{\frac{1}{n}+1}}{(2s)^{\frac{1}{n}}\cdot(m+1)^{\frac{1}{n}}}\ge$$

$$\overset{Mitrinovic}{\ge}\frac{3^{\frac{1}{n}+1}}{(3\sqrt{3}R)^{\frac{1}{n}}(m+1)^{\frac{1}{n}}}=\frac{3^{\frac{1}{n}+1-\frac{3}{2n}}}{\sqrt{(m+1)\cdot R}}=\frac{3^{1-\frac{1}{2n}}}{\sqrt{(m+1)R}}$$

$$\frac{1}{am+b}+\frac{1}{bm+c}+\frac{1}{cm+a}$$

$$=\frac{(bm+c)(cm+a)+(am+b)(cm+a)+(am+b)(bm+c)}{(am+b)(bm+c)(cm+a)}$$

$$\overset{Holder}{\le}\frac{\frac{[(bm+c)(cm+a)+(am+b)]^2}{3}}{\left[(abcm^3)^{\frac{1}{3}}+(abc)^{\frac{1}{3}}\right]^3}=\frac{(a+b+c)^2(m+1)^2}{(abc)(m+1)^3\cdot 3}$$

$$=\frac{4s^2}{3abc(m+1)}=\frac{4s^2}{3(4Rrs)(m+1)}=\frac{s}{3Rr(cm+1)}$$

$$\overset{Mitrinovic}{\le}\frac{3\sqrt{3}R}{2\cdot 3Rr(m+1)}=\frac{3^{\frac{1}{2}}}{(m+1)2r}\qquad (1)$$

$$\therefore\frac{1}{\sqrt[n]{am+b}}+\frac{1}{\sqrt[n]{bm+c}}+\frac{1}{\sqrt[n]{cm+a}}\overset{Holder}{\le}\sqrt[n]{(1+1+1)^{n-1}\sum\frac{1}{am+b}}$$

$$\le\sqrt[n]{3^{n-1}\cdot\frac{3^{\frac{1}{2}}}{(m+1)2r}}\text{, using (1)}=\sqrt[n]{\frac{3^{n-\frac{1}{2}}}{(m+1)2r}}=\frac{3^{1-\frac{1}{2n}}}{\sqrt[n]{(m+1)2r}}$$

Solution 3 by Marin Chirciu – Romania

RHS

$$\sum \frac{1}{\sqrt[n]{m\cdot a+b}} \overset{Holder}{\leq} \sqrt[n]{3^{n-1}\sum \frac{1}{m\cdot a+b}} \overset{(1)}{\leq} \sqrt[n]{3^{n-1}\cdot \frac{1}{m+1}\cdot \frac{\sqrt{3}}{2r}} = \frac{3^{1-\frac{1}{2n}}}{\sqrt[n]{(m+1)\cdot 2r}}$$

where (1) $\Leftrightarrow \sum \frac{1}{m\cdot a+b} \leq \frac{1}{m+1}\cdot \frac{\sqrt{3}}{2r}$, see:

$$\sum \frac{1}{m\cdot a+b} \overset{CS}{\leq} \frac{1}{(m+1)^2}\sum\left(\frac{m}{a}+\frac{1}{b}\right) = \frac{1}{(m+1)^2}(m+1)\sum \frac{1}{a} =$$

$$= \frac{1}{m+1}\sum \frac{1}{a} \overset{Leunberger}{\leq} \frac{1}{m+1}\cdot \frac{\sqrt{3}}{2r}$$

Equality holds if and only if the triangle is equilateral.

LHS

$$\sum \frac{1}{\sqrt[n]{m\cdot a+b}} \overset{Holder}{\geq} \frac{9}{\sum \sqrt[n]{m\cdot a+b}} \overset{(2)}{\geq} \frac{9}{\sqrt[n]{3^{n+\frac{1}{2}}(m+1)\cdot R}} = \frac{3^2}{3^{1+\frac{1}{2n}}\sqrt[n]{(m+1)\cdot R}} = \frac{3^{1-\frac{1}{2n}}}{\sqrt[n]{(m+1)\cdot R}},$$

Where (2) $\Leftrightarrow \sum \sqrt[n]{m\cdot a+b} \leq \sqrt[n]{3^{n+\frac{1}{2}}(m+1)\cdot R}$, see:

$$\sum \sqrt[n]{m\cdot a+b} \overset{Holder}{\leq} \sqrt[n]{3^{n-1}\sum(m\cdot a+b)} = \sqrt[n]{3^{n-1}(m+1)\sum a} =$$

$$= \sqrt[n]{3^{n-1}(m+1)\cdot 2p} \overset{Mitrinovic}{\leq}$$

$$\overset{Mitrinovic}{\leq} \sqrt[n]{3^{n-1}(m+1)\cdot 3\sqrt{3}R} = \sqrt[n]{3^{n+\frac{1}{2}}(m+1)\cdot R}$$

Equality holds if and only if the triangle is equilateral.

SP.531 If $a,b,c\geq 1$, then :

$$\sqrt{\frac{ab+bc+ca}{3}} - \sqrt[3]{abc} \geq \sqrt{\frac{\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}}{3}} - \frac{1}{\sqrt[3]{abc}}.$$

Proposed by Vasile Mircea Popa-Romania

Solution 1 by proposer

We write the inequality in the form $E(a,b,c)\geq 0$.
Without loss of generality, we may assume that $a\geq b\geq c\geq 1$.
We shall prove that: $E(a,b,c)\geq E(a,\sqrt{bc},\sqrt{bc})\geq 0$.
a) We will prove the inequality $E(a,b,c)\geq E(a,\sqrt{bc},\sqrt{bc})$.
The inequality can be written as follows:

$$\sqrt{\frac{ab+bc+ca}{3}}-\sqrt{\frac{bc+2a\sqrt{bc}}{3}}\geq\sqrt{\frac{a+b+c}{3abc}}-\sqrt{\frac{a+2\sqrt{bc}}{3abc}}$$

Or, equivalently:

$$\frac{\frac{1}{3}a\left(\sqrt{b}-\sqrt{c}\right)^2}{\sqrt{\frac{ab+bc+ca}{3}}+\sqrt{\frac{bc+2a\sqrt{bc}}{3}}}\geq\frac{\frac{1}{3}\frac{1}{abc}\left(\sqrt{b}-\sqrt{c}\right)^2}{\sqrt{\frac{a+b+c}{3abc}}+\sqrt{\frac{a+2\sqrt{bc}}{3abc}}}$$

To prove the inequality $E(a,b,c)\geq E(a,\sqrt{bc},\sqrt{bc})$ **it is enough to prove that:**

$$\left(\sqrt{b}-\sqrt{c}\right)^2\frac{1}{\sqrt{\frac{ab+bc+ca}{a^2}}+\sqrt{\frac{bc+2a\sqrt{bc}}{a^2}}}\geq\left(\sqrt{b}-\sqrt{c}\right)^2\frac{1}{\sqrt{abc\left(a+b+c\right)}+\sqrt{abc\left(a+2\sqrt{bc}\right)}}$$

For $b=c$ **this relationship it is true (case of equality).**
For $b\neq c$ **we will show that:**

$\frac{ab+bc+ca}{a^2}\leq abc\left(a+b+c\right)$ **(inequality 1) and** $\frac{bc+2a\sqrt{bc}}{a^2}\leq abc\left(a+2\sqrt{bc}\right)$ **(inequality 2)**

We prove inequalities (1) and (2). We have:

$ab+bc+ca\leq ab+a^2+ca=a\left(a+b+c\right)\leq a^3bc\left(a+b+c\right)$ **and:**

$$bc+2a\sqrt{bc}=a\left(\frac{bc}{a}+2\sqrt{bc}\right)\leq a\left(a+2\sqrt{bc}\right)\leq a^3bc\left(a+2\sqrt{bc}\right)$$

Thus, the inequality $E(a,b,c)\geq E(a,\sqrt{bc},\sqrt{bc})$ **is proved.**

b) We will prove the inequality: $E(a,\sqrt{bc},\sqrt{bc})\geq 0$ **. Let us denote:** $x=\sqrt{bc}$ **,** $a\geq x\geq 1$**.**

We have to show that:

$\sqrt{\frac{x^2+2ax}{3}}-\sqrt[3]{ax^2}\geq\sqrt{\frac{a+2x}{3ax^2}}-\frac{1}{\sqrt[3]{ax^2}}$ **. But, we have:** $x^2\geq 1$**.**

Since both sides of the above inequality are nonnegative, it is enough to prove the homogeneous inequality:

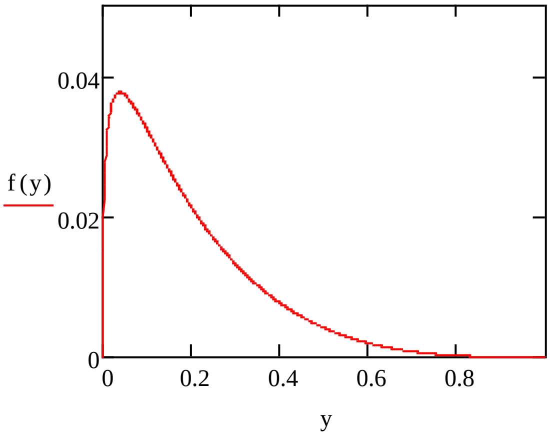
$\sqrt{\frac{x^2+2ax}{3}}-\sqrt[3]{ax^2}\geq x^2\left(\sqrt{\frac{a+2x}{3ax^2}}-\frac{1}{\sqrt[3]{ax^2}}\right)$**.This inequality is equivalently written:**

$\sqrt{\frac{1}{3}\left(y^2+2y\right)}-\sqrt[3]{y^2}\geq\sqrt{\frac{1}{3}\left(y^2+2y^3\right)}-\sqrt[3]{y^4}$ **, where:** $y=\frac{x}{a}$**,** $0<y\leq 1$

To prove this inequality,we study the variation and we draw the graph of the function:

$f(y)=\sqrt{\frac{1}{3}\left(y^2+2y\right)}-\sqrt[3]{y^2}-\sqrt{\frac{1}{3}\left(y^2+2y^3\right)}+\sqrt[3]{y^4}$ **,** $y\in\left(0,1\right]$

We obtain:



It follows that we have $f(y) \geq 0$, $y \in (0,1]$ (we have $f(1)=0$).Thus, the inequality $E(a,\sqrt{bc},\sqrt{bc}) \geq 0$ is proved.So, the inequality $E(a,b,c) \geq 0$ required in the statement of the problem is proved.

Remark: The expression:

$$M = \sqrt{\frac{ab+bc+ca}{3}}$$

is an elementary symmetric mean of the numbers a, b, c. We have: $M \geq G = \sqrt[3]{abc}$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

WLOG, we assume that $a \geq b \geq c$. Let $x := \sqrt{bc}$ and we will write the given inequality as

$$E(a,b,c) = \sqrt{\frac{ab+bc+ca}{3}} - \sqrt[3]{abc} - \sqrt{\frac{a+b+c}{3abc}} + \frac{1}{\sqrt[3]{abc}} \geq 0.$$

We will first prove that $E(a,b,c) \geq E(a,x,x)$. The inequality is successively equivalent to

$$\sqrt{\frac{ab+bc+ca}{3}} - \sqrt{\frac{a+b+c}{3abc}} \geq \sqrt{\frac{2a\sqrt{bc}+bc}{3}} - \sqrt{\frac{a+2\sqrt{bc}}{3abc}}$$

$$\left(\sqrt{ab+bc+ca} - \sqrt{2a\sqrt{bc}+bc}\right)\sqrt{abc} \geq \sqrt{a+b+c} - \sqrt{a+2\sqrt{bc}}$$

$$\frac{a\left(\sqrt{b}-\sqrt{c}\right)^2.\sqrt{abc}}{\sqrt{ab+bc+ca}+\sqrt{2a\sqrt{bc}+bc}} \geq \frac{\left(\sqrt{b}-\sqrt{c}\right)^2}{\sqrt{a+b+c}+\sqrt{a+2\sqrt{bc}}}$$

$$\sqrt{a^3bc(a+b+c)} + \sqrt{a^3bc\left(a+2\sqrt{bc}\right)} \geq \sqrt{ab+bc+ca} + \sqrt{2a\sqrt{bc}+bc},$$

which is true because $a^3bc(a+b+c)\geq ab+bc+ca$ and $a^3bc(a+2\sqrt{bc})\geq 2a\sqrt{bc}+bc$.

So it suffices to prove that $E(a,x,x)\geq 0$, and since $x\geq 1$, it suffices to prove that

$$\sqrt{\frac{2ax+x^2}{3}}-\sqrt[3]{ax^2}\geq x^2\left(\sqrt{\frac{a+2x}{3ax^2}}-\frac{1}{\sqrt[3]{ax^2}}\right).$$

Setting $t=\frac{a}{x}\geq 1$. The inequality is equivalent to

$$\sqrt{\frac{2t+1}{3}}-\sqrt[3]{t}\geq\sqrt{\frac{t+2}{3t}}-\frac{1}{\sqrt[3]{t}}\Leftrightarrow\sqrt{t(2t+1)}-\sqrt{t+2}\geq\sqrt{3}.\sqrt[6]{t}\left(\sqrt[3]{t^2}-1\right)$$

$$\Leftrightarrow\frac{2(t^2-1)}{\sqrt{t(2t+1)}+\sqrt{t+2}}\geq\sqrt{3}.\sqrt[6]{t}\left(\sqrt[3]{t^2}-1\right)$$

$$\overset{t\geq 1}{\Leftrightarrow}2\left(\sqrt[3]{t^4}+\sqrt[3]{t^2}+1\right)\geq\sqrt[6]{t}\left(\sqrt{3t(2t+1)}+\sqrt{3(t+2)}\right).$$

Let $y=\sqrt[6]{t}$. By AM − GM inequality, we have

$$RHS=\sqrt[6]{t}\left(\sqrt{3t(2t+1)}+\sqrt{3(t+2)}\right)\leq\sqrt[6]{t}\left(\frac{3t+(2t+1)}{2}+\frac{3+(t+2)}{2}\right)$$

$$=3y(y^6+1)\overset{?}{\leq}LHS$$

$$2(y^8+y^4+1)\geq 3y(y^6+1)\;\Leftrightarrow\;(y-1)^2(2y^6+y^5-y^3+y+1)\geq 0,$$

which is true and the proof is complete. Equality holds iff $a=b=c=1$.

SP.532 Prove that in any right triangle with the cathetus b and c we have the inequality:

$r\leq\frac{2-\sqrt{2}}{4}(b+c)$, where r is the inradii of the triangle.

Proposed by Laura Molea and Gheorghe Molea – Romania

Solution 1 by proposers

We have $r=\frac{S}{p}, S=\frac{bc}{2}\Rightarrow bc=r(a+b+c)\Rightarrow a=\frac{bc-rb-rc}{r}=\frac{bc}{r}-b-c.$

But $b^2+c^2=a^2\Rightarrow b^2+c^2=\left(\frac{bc}{r}-b-c\right)^2$

$$r^2b^2+r^2c^2=b^2c^2+b^2r^2+c^2r^2-2b^2cr-2bc^2r+2bcr^2$$

$$b^2c^2-2b^2cr-2bc^2r+2bcr^2=0|:bc$$

$$bc-2br-2cr+2r^2=0,\qquad b(c-2r)-2r(c-2r)=2r^2$$

$$(b-2r)(c-2r)=2r^2\quad(*)$$

But $a+b>c\Leftrightarrow a+b+c>2c\Leftrightarrow 1>\frac{2c}{a+b+c}$

$$\Leftrightarrow b>\frac{2bc}{a+b+c}=\frac{2r(a+b+c)}{a+b+c}=2r\Rightarrow b>2r$$

$\Rightarrow b-2r>0$. Analogous we obtain $c-2r>0$

As $Mg\le Ma\Rightarrow$ from inequality (*) the result

$$r\sqrt{2}=\sqrt{(b-2r)(c-2r)}\le\frac{b-2r+c-2r}{2}\Leftrightarrow 2r\sqrt{2}+4r\le b+c\Leftrightarrow$$

$$2r\left(\sqrt{2}+2\right)\le b+c\Leftrightarrow r\le\frac{1}{2\left(2+\sqrt{2}\right)}(b+c)\Leftrightarrow r\le\frac{2-\sqrt{2}}{4}(b+c)$$

We have equality $\Leftrightarrow$ the isosceles right triangle.

Solution 2 by Marin Chirciu – Romania

Using $r=\frac{b+c-a}{2}$ the inequality from enunciation can be written $\frac{b+c-a}{2}\le\frac{2-\sqrt{2}}{4}(b+c)\Leftrightarrow$

$$\Leftrightarrow 2a\ge\sqrt{2}(b+c)\Leftrightarrow 2a^2\ge(b+c)^2\Leftrightarrow 2(b^2+c^2)\ge(b+c)^2\Leftrightarrow(b-c)^2\ge 0$$

with equality for $b=c$.

Equality holds if and only if the triangle is right-angles isosceles.

SP.533 Prove that $k=\frac{4}{5}$ is the largest positive value of the constant k such that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}-5\ge k(a+b+c+d+e-5)$$

for any positive real numbers a,b,c,d,e satisfying

$$ab+bc+cd+de+ea=5$$

Proposed by Vasile Cîrtoaje – Romania

Solution by proposer

Setting: $a=x^2, b=e=\frac{m}{x^2}, c=d=\frac{1}{x}, m,x>0,$

from the equality constraint $ab+bc+cd+de+ea=5$, we get

$$2m+\frac{2m}{x^3}+\frac{1}{x^2}=5,\quad m=\frac{x(5x^2-1)}{2(x^3+1)},$$

and the desired inequality becomes

$$\frac{1}{x^2}+\frac{2x^2}{m}+2x-5\ge k\left(x^2+\frac{2m}{x^2}+\frac{2}{x}-5\right),$$

$$\frac{1}{x^2}+\frac{4x(x^3+1)}{5x^2-1}+2x-5\ge k\left(x^2+\frac{5x^2-1}{x^4+x}+\frac{2}{x}-5\right).$$

For $x\to\infty$, this inequality leads to the necessary condition $\frac{4}{5}\ge k$.

Further, we need to prove the inequality for $k=\frac{4}{5}$, i.e. to show that $E(a,b,c,d,e)\ge 0$,

where $E=5\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}\right)-4(a+b+c+d+e)-5.$

Let $T=(T_1,T_2,T_3,T_4,T_5)$ and $t=(t_1,t_2,t_3,t_4,t_5)$ be two decreasing sequences of nonnegative numbers. By Karamata majorization inequality applied to the convex function $f(x)=e^x$, if

$T_1\ge t_1, T_1T_2\ge t_1t_2, T_1T_2T_3\ge t_1t_2t_3, T_1T_2T_3T_4\ge t_1t_2t_3t_4$ and $T_1T_2T_3T_4T_5=t_1t_2t_3t_4t_5$, then

$$T_1+T_2+T_3+T_4+T_5\ge t_1+t_2+t_3+t_4+t_5.$$

Let (x_1,x_2,x_3,x_4,x_5) be a permutation of (a,b,c,d,e) such that $x_1\ge x_2\ge x_3\ge x_4\ge x_5$. According to Karamata's inequality, the largest cyclic sum of five terms x_ix_j, where $i\ne j$ and each x_i appears twice, is $S_1=x_1x_2+x_1x_3+x_2x_4+x_3x_5+x_4x_5$.

As a consequence, the smallest sum is

$$S_2=\sum_{1\le i<j\le 5}x_ix_j-S_1=x_1x_5+x_1x_4+x_2x_5+x_2x_3+x_3x_4.$$

Since $E(a,b,c,d,e)=E(x_1,x_2,x_3,x_4,x_5)$ and $ab+bc+cd+de+ea=5$ involves $S_2\le 5$, to prove that $E(a,b,c,d,e)\ge 0$ it suffices to show that $S_2\le 5$ involves $E(x_1,x_2,x_3,x_4,x_5)\ge 0$. Since increasing all the x_i by the same multiplicative factor increases the sum S_2 and decreases the function $E(x_1,x_2,x_3,x_4,x_5)$, we may consider $S_2=5$. So, we need to show that $E(a,b,c,d,e)\ge 0$ for

$$ae+ad+be+bc+cd=5,\quad a\ge b\ge c\ge d\ge e>0.$$

Denote: $x=\frac{a+b}{2}, y=\frac{d+e}{2}, a\ge x\ge b\ge c\ge d\ge y\ge e.$

Replacing a and e with $2x-b$ and $2y-d$, respectively, we have

$$5=a(d+e)+be+bc+cd=2(2x-b)y+b(2y-d)+bc+cd=$$
$$=4xy+bc-(b-c)d.$$

From this, we get: $5\geq 4xy+bc-(b-c)c=4xy+c^2$,

Hence $4xy\leq 5-c^2$, $c\leq\sqrt{5}$ and

$$5=4xy+bc-(b-c)d\leq 4xy+bc-(b-c)y=4xy+b(c-y)+cy$$
$$\leq 4xy+x(c-y)+cy=3xy+c(x+y)\leq\frac{3}{4}(5-c^2)+c(x+y),$$

hence: $4c(x+y)\geq 3c^2+5$

By the AM-HM inequality, we have

$$\frac{1}{a}+\frac{1}{b}\geq\frac{4}{a+b}=\frac{2}{x},\quad \frac{1}{d}+\frac{1}{e}\geq\frac{2}{y}.$$

Thus, it suffices to show that the conditions

$$4xy\leq 5-c^2,\quad 4c(x+y)\geq 3c^2+5,\quad x\geq c\geq y>0$$

involve

$$5\left(\frac{2}{x}+\frac{2}{y}+\frac{1}{c}\right)\geq 4(2x+2y+c)+5,$$

that is

$$2(x+y)\left(\frac{5}{xy}-4\right)+\frac{5}{c}-4c-5\geq 0.$$

Since

$$\frac{5}{xy}-4\geq\frac{20}{5-c^2}-4=\frac{4c^2}{5-c^2},$$

it suffices to show that: $\frac{8(x+y)c^2}{5-c^2}+\frac{5}{c}-4c-5\geq 0.$

Indeed,

$$\frac{8(x+y)c^2}{5-c^2}+\frac{5}{c}-4c-5\geq\frac{2c(3c^2+5)}{5-c^2}+\frac{5}{c}-4c-5$$
$$=\frac{5(2c^4+c^3-3c^2-5c+5)}{c(5-c^2)}=\frac{5(c-1)^2(2c^2+5c+5)}{c(5-c^2)}\geq 0$$

The proof is completed. For $k=\frac{4}{5}$, the equality occurs when $a=b=c=d=e=1$.

SP.534 If the lengths a, b, c of the sides of a triangle are the roots of the equation $kx^3 - lx^2 + 9kx - l = 0 \quad (k \cdot l \neq 0)$, then find the area of the triangle.

Proposed by George Apostolopoulos – Greece

Solution 1 by proposer

From the Cartan – Viete formula's, we have $a + b + c = \frac{l}{k}, ab + bc + ca = 9, abc = \frac{l}{k}$. So

$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{ab+bc+ca}{abc} = \frac{9}{\frac{l}{k}} = \frac{9k}{l}$. Now, we have $\frac{a+b+c}{3} = \frac{l}{3k}$, and

$\frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} = \frac{3}{\frac{9k}{l}} = \frac{l}{3k}$, namely $\frac{a+b+c}{3} = \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$. So, the result follows immediately from the fact that the AM-HM inequality becomes equality iff $a = b = c$. Namely the triangle is equilateral. From $ab + bc + ca = 9$, we have $a^2 = 3$, and the area $= \frac{a^2\sqrt{3}}{4} = \frac{3\sqrt{3}}{4}$.

Solution 2 by Martin Celli-Mexico

For $x = a, b, c$ we have

$$kx(x^2 + 9) = l(x^2 + 1),$$

$$F(x) = \ln\left(\frac{l}{k}\right), \text{ where } F(x) = \ln\left(\frac{x(x^2+9)}{x^2+1}\right) \quad (x > 0).$$

We can easily obtain the following expression:

$$F'(x) = \frac{(x^2-3)^2}{x(x^2+9)(x^2+1)} > 0 \text{ for } x \neq \sqrt{3}.$$

Thus, the function F is strictly increasing. As $F(a) = F(b) = F(c)$, we have $a = b = c$: the triangle is equilateral, its area is $a^2\sqrt{3}/4$. On the other hand, we have

$$x^3 - 3ax^2 + 3a^2x - a^3 = (x - a)^3 = (x - a)(x - b)(x - c) = x^3 - \frac{l}{k}x^2 + 9x - \frac{l}{k}$$

So $3a^2 = 9$, the area of the triangle is $3\sqrt{3}/4$.

Solution 3 by Marin Chirciu – Romania

By Viete relationships we have $a + b + c = \frac{l}{k}, ab + bc + ca = 9, abc = \frac{l}{k}$

Using the identity in triangle $16S^2 = 2\sum b^2c^2 - \sum a^4$.

We have $\sum b^2c^2 = 81 - \frac{2l^2}{k^2}$ and $\sum a^4 = \frac{l^4}{k^4} - \frac{32l^2}{k^2} + 162$ which follows from:

$$\sum b^2c^2 = \left(\sum bc\right)^2 - 2abc\sum a = 9^2 - 2\cdot\frac{l}{k}\cdot\frac{l}{k} = 81 - \frac{2l^2}{k^2}$$

$$(a+b+c)^4 = \sum a^4 + 6\sum b^2c^2 + 4\sum bc\,(b^2+c^2) + 12abc\sum a$$

$$\sum bc(b^2+c^2) = \sum a^2\sum bc - abc\sum a,$$

$$\sum bc(b^2+c^2) = \left(\frac{l^2}{k^2} - 18\right)\cdot 9 - \frac{l^2}{k^2} = \frac{8l^2}{k^2} - 162$$

$$\sum a^2 = \left(\sum a\right)^2 - 2\sum bc = \frac{l^2}{k^2} - 2\cdot 9 = \frac{l^2}{k^2} - 18$$

$$\frac{l^4}{k^4} = \sum a^4 + 6\left(81 - \frac{2l^2}{k^2}\right) + 4\left(\frac{8l^2}{k^2} - 162\right) + \frac{12l^2}{k^2} =$$

$$= \sum a^4 + 486 - \frac{12l^2}{k^2} + \frac{32l^2}{k^2} - 648 + \frac{12l^2}{k^2} =$$

$$= \frac{32l^2}{k^2} - 162 \Rightarrow \sum a^4 = \frac{l^4}{k^4} - \frac{32l^2}{k^2} + 162$$

We obtain $16S^2 = 2\sum b^2c^2 - \sum a^4 =$

$$= 2\left(81 - \frac{2l^2}{k^2}\right) - \left(\frac{l^4}{k^4} - \frac{32l^2}{k^2} + 162\right) = \frac{28l^2}{k^2} - \frac{l^4}{k^3} \Rightarrow$$

$$\Rightarrow 16S^2 = \frac{28l^2}{k^2} - \frac{l^4}{k^4} \Leftrightarrow S^2 = \frac{1}{16}\left(\frac{28l^2}{k^2} - \frac{l^4}{k^4}\right) \Leftrightarrow S = \frac{1}{4}\sqrt{\frac{28l^2}{k^2} - \frac{l^4}{k^4}}$$

It is necessary the condition $\frac{28l^2}{k^2} - \frac{l^4}{k^4} > 0 \Leftrightarrow \frac{1}{k} < 2\sqrt{7}$

SP.535 Determine all the numbers $\overline{abcd}$ such that:

$$1 + a + b + c + a\cdot b + b\cdot c + c\cdot a = a\cdot b\cdot c\cdot d$$

Proposed by Neculai Stanciu, Titu Zvonaru – Romania

Solution 1 by proposers

$$dabc - ab - bc - ac - a - b = c + 1 \Leftrightarrow (dc-1)ab - (c+1)a - (c+1)b = c + 1 \quad (1)$$

If $dc = 1$, then $d = c = 1$ so (1) doesn't solutions. Multiply (1) with $dc - 1$, then

$$(dc-1)^2ab - (dc-1)(c+1)a - (dc-1)(c+1)b = (dc-1)(c+1) \Leftrightarrow$$

$\Leftrightarrow [(dc-1)a-(c+1)][(dc-1)b-(c+1)] = (c+1)^2 + (dc-1)(c+1) \Leftrightarrow$

$\Leftrightarrow [(dc-1)a-(c+1)][(dc-1)b-(c+1)] = (d+1)c(c+1)$ **(*)**

WLOG we can assume that $a \geq b \geq c$. Then

$(dc-1)a-(c+1) \geq dc^2-2c-1, (dc-1)b-(c+1) \geq dc^2-2c-1$

We have the cases:

1. If $d \geq 3$, then $dc^2-2c-1 \geq 3c^2-2c-1 = (c-1)(3c+1) \geq 0$ and by (*) yields

$(d+1)c(c+1) = [(dc-1)a-(c+1)][(dc-1)a-(c+1)] \geq (dc^2-2c-1)^2 \Leftrightarrow$

$\Leftrightarrow d^2c^4+4c^2+1-4dc^3-2dc^2+4c-(d+1)c^2-(d+1)c \leq 0 \Leftrightarrow$

$\Leftrightarrow d^2c^4-4dc^3-3(d-1)c^2-(d-1)c+1 \leq 0$ **(2)**

If $d \geq 8$, then

$d^2c^4-4dc^3-3(d-1)c^2-(d-3)c+1 \geq 8dc^4-4dc^3-3dc^2-dc+3c^2+3c+1 =$

$= 4dc^3(c-1)+3dc^2(c^2-1)+dc(c^3-1)+3c^2+3c+1 > 0$ **so (2) is false..**

1.1 $d = 7P$: (2) becomes $49c^4-28c^3-18c^2-4c+1 \leq 0 \Leftrightarrow$

$\Leftrightarrow (c-1)(49c^3+21c^2+3c-1) \leq 0$ **and because**

$49c^3+21c^2+3c-1 > 0$**, we have $c = 1$; (*) becomes**

$(6a-2)(6b-2)+16 \Leftrightarrow (3a-1)(3b-1) = 4$ **and we obtain $a = b = c = 1$.**

1.2. $d = 6$: (2) becomes $36c^4-24c^3-15c^2-3c+1 \leq 0$

For $c \geq 2$: $36c^4-24c^3-15c^2-3c+1 \geq 72c^3-24c^3-15c^2-3c+1 =$

$= 48c^3-15c^2-3c+1 = 30c^3+15c^2(c-1)+3c(c^2-1)+1 > 0$ **and results**

$c = 1$; (*) becomes $(5a-2)(5b-2) = 12$ and we not obtain solutions.

1.3. $d = 5$: (2) becomes $25c^4-20c^3-12c^2-2c+1 \leq 0$

For $c \geq 2$ we have $25c^4-20c^3-12c^2-2c+1 \geq 50c^3-20c^3-12c^2-2c+1 =$

$= 30c^3-12c^2-2c+1 = 16c^3+12c^2(c-1)+2c(c^2-1)+1 > 0$**, so**

$c = 1$; (*) becomes

$(4a-2)(4b-2) = 12 \Leftrightarrow (2a-1)(2b-1) = 3$ **and we obtain the solution**

$a = 2, b = 1, c = 1$

1.4. $d = 4$: (2) becomes $16c^4-16c^3-9c^2-c+1 \leq 0$. For $c \geq 2$ we have

$16c^4-16c^3-9c^2-c+1 \leq 32c^3-16c^3-9c^2-c+1 =$

$= 16c^3-9c^2-c+1 = 6c^3+9c^2(c-1)+c(c^2-1)+1 > 0$ **and $c = 1$; (*) becomes**

$(3a-2)(3b-2) = 10$ **and we obtain solution $a = 4, b = 1, c = 1$.**

1.5 $d = 3$; (2) becomes $9c^4-12c^3-6c^2+1 \leq 0$. For $c \geq 2$ we have

$9c^4-12c^3-6c^2+1 \geq 18c^3-12c^3-6c^2+1 = 6c^3-6c^2+1 = 6c^2(c-1)+1 >$ 0 **so $c = 1$; (*) becomes $(2a-2)(2b-2) = 8 \Leftrightarrow (a-1)(b-1) = 2$ and we obtain**

$a = 3, b = 2, c+1$.

2. $d = 2$

2.1. For $c = 1$, (*) is $(a-2)(b-2) = 6$ and we obtain the solutions $a = 8, b = 3, c = 1$ and $a = 5, b = 4, c = 1$.

2.2. For $c \geq 2$, $2c^2-2c-1 \geq 4c-2c-1 = 2c-1 > 0$ and (2) is true, i.e.

$4c^4-8c^3-6c^2+c+1 \leq 0$. But $c \geq 3$ so,

$4c^4-8c^3-6c^2+c+1 \geq 12c^3-8c^3-6c^2+c+1 = 4c^3-6c^2+c+1 =$

$= 2c^3+2c^2(c-3)+c+1 > 0$**, so $c = 2$; (*) becomes**

$(3a-3)(3b-3) = 18 \Leftrightarrow (a-1)(b-1) = 2$**, hence $a = 3, b = 2, c = 2$.**

3. $d = 1$**. Since** $c \neq 1$**, we have:**

3.1.For $c = 2$**, (*) becomes** $(a-3)(b-3) = 12$ **and we obtain**

$(a = 15, b = 4, c = 2), (a = 9, b = 5, c = 2)$ **and** $(a = 7, b = 6, c = 2)$**.**

3.2 For $c = 3$**, (*) becomes** $(2a-4)(2b-4) = 24 \Leftrightarrow (a-2)(b-2) = 6$ **and we obtain**

$(a = 8, b = 3, c = 3)$ **and** $(a = 5, b = 4, c = 3)$**.**

3.3. For $c \geq 4$ **we have** $c^2 - 2c - 1 \geq 4c - 2c - 1 = 2c - 1 > 0$ **so (2), i.e.**

$c^4 - 4c^3 + 2c + 1 \leq 0$**, true**

$c^4 - 4c^3 + 2c + 1 = c^4(c-4) + 2c + 1 > 0$**, we not obtain solutions.**

In conclusion,

a	b	c	d
1	1	1	7
2	1	1	5
4	1	1	4
3	2	1	2
5	4	1	2
8	3	1	2
3	2	2	2
15	4	2	1
9	5	2	1
7	6	2	1
8	3	3	1
5	4	3	1

Hence $\overline{abcd} \in \{1117, 2115, 3213, 3222, 4114, 5412, 5431, 7621, 8312, 8331, 9521\}$

Solution 2 by Marin Chirciu – Romania

Necessarily a, b, c, d **are non-zero digits.**

For $a = 1 \Rightarrow 2 + 2b + 2c + bc = bcd \Leftrightarrow 2(1 + b + c) = bc(d-1)$**;**

For $b = 1$ **agreed** $c = 1, 2, 4 \Rightarrow d = 7, 5, 4$**. It follows** $\overline{abcd} = 1117, 1125, 1144$**.**

For $b = 2$ **agreed** $c = 1, 3 \Rightarrow d = 5, 3$**. It follows** $\overline{abcd} = 1215, 1233$**.**

For $b = 3$ **agreed** $c = 2, 8 \Rightarrow d = 3, 2$**. It follows** $\overline{abcd} = 1323, 1382$**.**

For $b = 4$ **agreed** $c = 1, 5 \Rightarrow d = 4, 2$**. It follows** $\overline{abcd} = 1414, 1452$**.**

For $b = 5$ **agreed** $c = 4 \Rightarrow d = 2$**. It follows** $\overline{abcd} = 1542$**.**

For $b = 6, 7$ **we don't have solutions.**

For $b = 8$ **agreed** $c = 3 \Rightarrow d = 2$**. It follows** $\overline{abcd} = 1832$**.**

For $b = 9$ **we don't have solutions.**

For $a = 2 \Rightarrow 3 + 3b + 3c + bc = 2bcd \Leftrightarrow 3(1 + b + c) = bc(2d-1)$**;**

For $b = 1$ **agreed** $c = 1, 3 \Rightarrow d = 5, 3$**. It follows** $\overline{abcd} = 2115, 2133$**.**

For $b = 2$ **agreed** $c = 3 \Rightarrow d = 2$**. It follows** $\overline{abcd} = 2232$**.**

For $b = 3$ **agreed** $c = 1, 2 \Rightarrow d = 3, 2$**. It follows** $\overline{abcd} = 2313, 2322$**.**

For $b=4$ we don't have solutions.
For $b=5$ agreed $c=9\Rightarrow d=1$. It follows $\overline{abcd}=2591$.
For $b=6$ agreed $c=7\Rightarrow d=1$. It follows $\overline{abcd}=2671$.
For $b=7,8$ we don't have solutions.
For $b=9$ agreed $c=5\Rightarrow d=2$. It follows $\overline{abcd}=2951$.
For $a=3\Rightarrow 4+4b+4c+bc=3bcd\Leftrightarrow 4(1+b+c)=bc(3d-1)$;
For $b=1$ agreed $c=2,8\Rightarrow d=3,2$. It follows $\overline{abcd}=3123,3182$.
For $b=2$ agreed $c=1,2\Rightarrow d=3,2$. It follows $\overline{abcd}=3213,3222$.
For $b=3$ agreed $c=8\Rightarrow d=1$. It follows $\overline{abcd}=3381$.
For $b=4$ agreed $c=5\Rightarrow d=1$. It follows $\overline{abcd}=3451$.
For $b=5$ agreed $c=4\Rightarrow d=1$. It follows $\overline{abcd}=3541$.
For $b=6,7,8,9$ we don't have solutions.
For $a=4\Rightarrow 5+5b+5c+bc=4bcd\Leftrightarrow 5(1+b+c)=bc(4d-1)$;
For $b=1$ agreed $c=1,5\Rightarrow d=4,2$. It follows $\overline{abcd}=4114,4152$.
For $b=2$ we don't have solutions.
For $b=3$ agreed $c=5\Rightarrow d=1$. It follows $\overline{abcd}=4351$.
For $b=4$ we don't have solutions.
For $b=5$ agreed $c=1,3\Rightarrow d=2,1$. It follows $\overline{abcd}=4512,4531$.
For $b=6,7,8,9$ we don't have solutions.
For $a=5\Rightarrow 6+6b+6c+bc=5bcd\Leftrightarrow 6(1+b+c)=bc(5d-1)$;
For $b=1$ agreed $c=4\Rightarrow d=2$. It follows $\overline{abcd}=5142$.
For $b=2,3$ we don't have solutions.
For $b=4$ agreed $c=1,3\Rightarrow d=2,1$. It follows $\overline{abcd}=5412,5431$.
For $b=5,6,7,8,9$ we don't have solutions.
For $a=6\Rightarrow 7+7b+7c+bc=6bcd\Leftrightarrow 7(1+b+c)=bc(6d-1)$;
For $b=1$ we don't have solutions.
For $b=2$ agreed $c=7\Rightarrow d=1$. It follows $\overline{abcd}=6271$.
For $b=3,4,5,6$ we don't have solutions.
For $b=7$ agreed $c=2\Rightarrow d=1$. It follows $\overline{abcd}=6721$.
For $b=8,9$ we don't have solutions.
For $a=7\Rightarrow 8+8b+8c+bc=7bcd\Leftrightarrow 8(1+b+c)=bc(7d-1)$;
For $b=1$ we don't have solutions.
For $b=2$ agreed $c=6\Rightarrow d=1$. It follows $\overline{abcd}=7261$.
For $b=3,4,5$ we don't have solutions.
For $b=6$ agreed $c=2\Rightarrow d=1$. It follows $\overline{abcd}=7621$.
For $b=7,8,9$ we don't have solutions.
For $a=8\Rightarrow 9+9b+9c+bc=8bcd\Leftrightarrow 9(1+b+c)=bc(8d-1)$;
For $b=1$ agreed $c=3\Rightarrow d=2$. It follows $\overline{abcd}=8132$.
For $b=2$ we don't have solutions.

For $b=3$ agreed $c=1,3\Rightarrow d=2,1$. It follows $\overline{abcd}=8312,8331$.
For $b=4,5,6,7,8,9$ we don't have solutions.
For $a=9\Rightarrow 10+10b+10c+bc=9bcd\Leftrightarrow 10(1+b+c)=bc(9d-1)$;
For $b=1$ we don't have solutions.
For $b=2$ agreed $c=5\Rightarrow d=1$. It follows $\overline{abcd}=9251$.
For $b=3,4$ we don't have solutions.
For $b=5$ agreed $c=2\Rightarrow d=1$. It follows $\overline{abcd}=9521$.
For $b=6,7,8,9$ we don't have solutions.
Finally, we deduce:
$\overline{abcd}$
$=1117,1125,1144,1215,1233,1223,1382,1414,1452,1542,1832,2115,2133,2232,$
$2313,2322,2591,2671,2951,3123,3182,3213,3222,3381,3151,3541,4114,4152,$
$4351,4512,4531,5142,5412,5431,6271,6721,7261,7621,8132,8312,8331,9251,9251$

SP.536 If $x\geq 0$ then:

$$\frac{2}{\sqrt{\pi}}\left(\int_0^x e^{-t^2}\,dt\right)^2+\int_0^{2x}e^{-t^2}\,dt\geq 2\int_0^x e^{-t^2}\,dt$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ahmed Salem-Tunisia

$$\text{Let } Q(x)=\frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-\frac{v^2}{2}}\,dv$$

Prob. that standard Gaussian RVX exceeds x

$$\int_0^x e^{-t^2}\,dt\overset{u=\sqrt{2}t}{=}\sqrt{2\pi}\frac{1}{\sqrt{2\pi}}\int_0^{\sqrt{2}x}e^{-\frac{u^2}{2}}\frac{1}{\sqrt{2}}\,du=\sqrt{\pi}\frac{1}{\sqrt{2\pi}}\int_0^{\sqrt{2}x}e^{-\frac{u^2}{2}}\,du$$

$$=\sqrt{\pi}\left(\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{u^2}{2}}\,du-\frac{1}{\sqrt{2\pi}}\int_{\sqrt{2}x}^\infty e^{-\frac{u^2}{2}}\,du\right)=\sqrt{\pi}\left(\frac{1}{2}-Q(\sqrt{2}x)\right)$$

The given inequality is equivalent to

$$\frac{2}{\sqrt{\pi}}\cdot\pi\left(\frac{1}{2}-Q(\sqrt{2}x)\right)^2+\sqrt{\pi}\left(\frac{1}{2}-Q(2\sqrt{2}x)\right)\geq 2\sqrt{\pi}\left(\frac{1}{2}-Q(\sqrt{2}x)\right)$$

$$\Rightarrow\frac{\sqrt{\pi}}{2}+2\sqrt{\pi}Q^2(\sqrt{2}x)-2\sqrt{\pi}Q(\sqrt{2}x)+\frac{\sqrt{\pi}}{2}-\sqrt{\pi}Q(2\sqrt{2}x)\geq\sqrt{\pi}-2\sqrt{\pi}Q(\sqrt{2}x)$$

$$\Rightarrow 2Q^2(\sqrt{2}x)\geq Q(2\sqrt{2}x)$$

$$Q(x)=\frac{1}{2\pi}\int_x^{\infty}\int_{-\infty}^{\infty}e^{-\frac{1}{2}(u^2+v^2)}\,dudv=\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\int_{x\sec\phi}^{\infty}-d\,e^{-\frac{1}{2}r^2}\,d\phi=$$

$$=\frac{1}{\pi}\int_0^{\frac{\pi}{2}}e^{-\frac{1}{2}x^2\sec^2\phi}\,d\phi\overset{\phi\to\frac{\pi}{2}-\phi}{=}\frac{1}{\pi}\int_0^{\frac{\pi}{2}}e^{-\frac{x^2}{2\sin^2\phi}}\,d\phi$$

$$\left(1-2Q(x)\right)^2=\frac{1}{2\pi}\int_{-x}^{x}\int_{x}^{x}e^{-\frac{1}{2}(u^2+v^2)}\,du\,dv=\frac{4}{\pi}\int_0^{\frac{\pi}{4}}\int_0^{x\sec\phi}-de^{-\frac{1}{2}r^2}\,d\phi$$

$$=\frac{4}{\pi}\int_0^{\frac{\pi}{4}}\left(1-e^{-\frac{1}{2}x^2\sec^2\phi}\right)d\phi=1-4Q(x)+4Q^2(x)$$

$$\Rightarrow Q^2(x)=Q(x)-\frac{1}{\pi}\int_0^{\frac{\pi}{4}}e^{-\frac{1}{2}x^2\sec^2\phi}=\frac{1}{\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}e^{-\frac{1}{2}x^2\sec^2\phi}\,d\phi\overset{\phi\to\frac{\pi}{2}-\phi}{=}\frac{1}{2}\int_0^{\frac{\pi}{4}}e^{-\frac{x^2}{2\sin^2\phi}}\,d\phi$$

$$2Q^2(\sqrt{2}x)=\frac{2}{\pi}\int_0^{\frac{\pi}{4}}e^{-\frac{x^2}{\sin\phi}}\,d\phi$$

$$Q(2\sqrt{2}x)=\frac{1}{\pi}\int_0^{\frac{\pi}{2}}e^{-\frac{4x^2}{\sin B}}\,dB\overset{\phi=\frac{B}{2}}{=}\frac{2}{\pi}\int_0^{\frac{\pi}{4}}e^{-\frac{4x^2}{\sin^2(2\phi)}}\,d\phi$$

$$2Q^2(\sqrt{2}x)-Q(2\sqrt{2}x)=\frac{2}{\pi}\int_0^{\frac{\pi}{4}}\left(e^{-\frac{x^2}{\sin^2\phi}}-e^{-\frac{4x^2}{\sin^2(2\phi)}}\right)d\phi$$

For $\phi\in\left(0,\frac{\pi}{4}\right]$:

$$4\sin^2\phi>4\sin^2\phi\cos^2\phi=\sin^2 2\phi\Rightarrow\frac{4}{\sin^2 2\phi}>\frac{1}{\sin^2\phi}\Rightarrow-\frac{x^2}{\sin^2\phi}>-\frac{4x^2}{\sin^2 2\phi}\Rightarrow$$

$$\Rightarrow e^{-\frac{x^2}{\sin^2\phi}}>e^{-\frac{4x^2}{\sin^2(2\phi)}}$$

$$\Rightarrow 2Q^2(\sqrt{2}x)\ge Q(2\sqrt{2}x),\frac{2}{\sqrt{\pi}}\left(\int_0^x e^{-t^2}dt\right)^2+\int_0^{2x}e-t^2\,dt\ge 2\int_0^x e^{-t^2}\,dt$$

Solution 2 by Djamel Arrouche-Algeria

$$\frac{2}{\sqrt{\pi}}\left(\int_0^x e^{-t^2}dt\right)^2+\int_0^{2x}e^{-t^2}\,dt\ge 2\int_0^x e^{-t^2}\ \ldots\ \text{E}$$

$$\text{erf}(y)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$$

$$E\Leftrightarrow\text{erf}^2(x)+\text{erf}(2x)\ge 2\,\text{erf}(x)$$

Proposition $\forall(x,y)\in[0,\infty[^2$: $\operatorname{erf}(x)\operatorname{erf}(y)+\operatorname{erf}(x+y)\geq\operatorname{erf}(x)+\operatorname{erf}(y)$

Let $y\in\mathbb{R}_+$ $\forall x\geq 0$

$$f_y(x)=\operatorname{erf}(x)\operatorname{erf}(y)+\operatorname{erf}(x+y)-\operatorname{erf}(x)-\operatorname{erf}(y)$$

$$\Rightarrow f_y'(x)=\frac{2}{\sqrt{\pi}}\left(e^{-x^2}\operatorname{erf}(y)+e^{-(x+y)^2}-e^{-x^2}\right)$$

$$\Leftrightarrow\frac{\sqrt{\pi}}{2}e^{-x^2}f'(x)=\operatorname{erf}(y)+e^{-y^2-2xy}-1$$

$$f_y'(x)\geq 0\Leftrightarrow e^{-y^2-2xy}\geq 1-\operatorname{erf}(y)\Rightarrow x\leq\frac{1}{-2y}(\ln(1-\operatorname{erf}(y))+y)=a_y$$

$$1-\operatorname{erf}(y)=\frac{2}{\sqrt{\pi}}\int_y^{\infty}e^{-t^2}\,dt;t=y+z\Rightarrow\frac{2}{\sqrt{\pi}}e^{-y^2}\int_0^{\infty}e^{-z^{2-yz}}\,dz\leq e^{-y^2}$$

$$a_y\geq\frac{1}{-2y}(-2x+y)=\frac{1}{2}\Rightarrow a_y\in[0,+\infty[$$

$$f_y(x)\leq f\left(a_y\right)$$

$$\Rightarrow f_y(x)=\min\left(f_y(0),\lim_{x\to\infty}f_y(x)\right)\Rightarrow f_y(0)=0$$

$$\lim_{y\to\infty}f_y(x)=\operatorname{erf}(y)+1-1-\operatorname{erf}(y)=0$$

$f_y(x)\geq 0\Rightarrow f_y(x)\geq 0;\forall x\in[0,\infty[\Rightarrow$ since y is arbitrary $\Rightarrow\forall(x,y)\in\mathbb{R}_+^2$

$$\operatorname{erf}(x)\operatorname{erf}(y)+\operatorname{erf}(x+y)\geq\operatorname{erf}(x)+\operatorname{erf}(y)$$

$$c=y\Rightarrow\operatorname{erf}^2(x)+\operatorname{erf}(2x)\geq 2\operatorname{erf}(x)$$

Solution 3 by Hikmat Mammadov-Azerbaijan

Results:

$$\int_0^{+\infty}e^{-t^2}\,dt=\frac{\sqrt{\pi}}{2}$$

$$\int_x^{+\infty}e^{-t^2}\,dt=\frac{\sqrt{\pi}}{2}-\int_0^x e^{-t^2}\,dt;\text{ for }x\geq 0$$

Proof: To simplify let pose: $a=\int_0^x e^{-t^2}\,dt$ and $b=\int_0^{2x}e^{-t^2}\,dt$

The inequality is equivalent to: $\frac{2}{\sqrt{\pi}}a^2+b\geq 2a\Leftrightarrow a^2+\frac{\sqrt{\pi}}{2}b\geq\sqrt{\pi}a$

$$\Leftrightarrow a^2-\sqrt{\pi}a\geq\frac{\sqrt{\pi}}{2}b\Leftrightarrow\left(a-\frac{\sqrt{\pi}}{2}\right)^2\geq\frac{\pi}{4}-\frac{\sqrt{\pi}}{2}b$$

$$\Leftrightarrow\left(a-\frac{\sqrt{\pi}}{2}\right)^2\geq\frac{\sqrt{\pi}}{2}\left(\frac{\sqrt{\pi}}{2}-b\right)\Leftrightarrow\frac{\left(a-\frac{\sqrt{\pi}}{2}\right)^2}{\left(\frac{\sqrt{\pi}}{2}-b\right)}\geq\frac{\sqrt{\pi}}{2}$$

This is equivalent to:

$$\frac{\left(\int_x^{+\infty}e^{-t^2}dt\right)^2}{\int_{2x}^{+\infty}e^{-t^2}dt}\geq\frac{\sqrt{\pi}}{2}\qquad (*)$$

Let's prove (*): for that we consider the function $\varphi(x)=\frac{\left(\int_x^{+\infty}e^{-t^2}\right)^2}{\int_{2x}^{+\infty}e^{-t^2}dt}$

We have:

$$\varphi'(x)=\frac{-2e^{-x^2}\int_x^{+\infty}e^{-t^2}dt\left(\int_{2x}^{+\infty}e^{-t^2}dt\right)+2e^{-4x^2}\left(\int_x^{+\infty}e^{-t^2}dt\right)^2}{\int_{2x}^{+\infty}e^{-t^2}dt}$$

$$=\frac{2e^{-x^2}\left(\int_x^{+\infty}e^{-t^2}dt\right)\left(e^{-3x^2}\int_x^{+\infty}e^{-t^2}dt-\int_{2x}^{+\infty}e^{-t^2}dt\right)}{\left(\int_{2x}^{+\infty}e^{-t^2}dt\right)^2}$$

Now we should find the sign of the expression:

$$e^{-3x^2}\int_x^{+\infty}e^{-t^2}dt-\int_{2x}^{+\infty}e^{-t^2}dt$$

Notice that:

$$\int_{2x}^{+\infty}e^{-t^2}dt=\int_x^{+\infty}e^{-(t+x)^2}dt$$

We know that: $(t+x)^2=t^2+2tx+x^2$**; and for** $t\geq x$ **we obtain:**

$(t+x)^2\geq t^2+3x^2$ **witch gives** $-(t+x)^2\leq-(t^2+3x^2)$

Then:

$$\int_x^{+\infty}e^{-(t+x)^2}dt\leq\int_x^{+\infty}e^{-(t^2+3x^2)}dt\Rightarrow\int_{2x}^{+\infty}e^{-t^2}dt\leq e^{-3x^2}\int_x^{+\infty}e^{-t^2}dt$$

So:

$e^{-3x^2}\int_x^{+\infty}e^{-t^2}dt-\int_{2x}^{+\infty}e^{-t^2}dt\geq 0$**; for** $x\geq 0$

Conclusion: we have $\varphi'(x)\geq 0$ **for** $x\geq 0$ **which means** φ **is increasing function**

Then: $\varphi(x) \geq \varphi(0)$ for $x \geq 0$; and $\varphi(0) = \frac{\left(\int_0^{+\infty} e^{-t^2} dt\right)^2}{\int_0^{+\infty} e^{-t^2} dt} = \int_0^{+\infty} e^{-t^2}\, dt = \frac{\sqrt{\pi}}{2}$

Finally we get:

$$\frac{\left(\int_x^{+\infty} e^{-t^2}\, dt\right)^2}{\int_{2x}^{+\infty} e^{-t^2}\, dt} \geq \frac{\sqrt{\pi}}{2}$$

SP.537 Solve for real numbers:

$$3e^x + 3e^{3x} + 1 = 4e^{2x} + 3 \cdot \sqrt[3]{e^{4x}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$3e^x + 3e^{3x} + 1 = (2e^x + 2e^{3x}) + (e^x + e^{3x} + 1) =$$

$$= 2(e^x + e^{3x}) + (e^x + e^{3x} + 1) \overset{AM-GM}{\geq} 2 \cdot 2\sqrt{e^x \cdot e^{3x}} + 3 \cdot \sqrt[3]{e^x \cdot e^{3x} \cdot 1} =$$

$$= 4\sqrt{e^{4x}} + 3\sqrt[3]{e^{4x}} = 4e^{2x} + 3\sqrt[3]{e^{4x}}$$

Equality holds for: $e^x = e^{3x}$ and $e^x = e^{3x} = 1$. Solution: $x = 0$

Solution 2 by Marin Chiricu – Romania

We have $3e^x + 3e^{3x} + 1 = 4e^{2x} + 3\sqrt[3]{e^{4x}} \Leftrightarrow 3e^x + 3e^{3x} + 1 = 4e^{2x} + 3e^{\frac{4x}{3}}$

With the substitution $e^{\frac{x}{3}} = t > 0$ the equation becomes

$$3t^3 + 3t^9 + 1 = 4t^6 + 3t^4 \Leftrightarrow 3t^9 - 4t^6 - 3t^4 + 3t^3 + 1 = 0 \Leftrightarrow$$

$$\Leftrightarrow (t-1)^2(3t^7 + 6t^6 + 9t^5 + 8t^4 + 7t^3 + 3t^2 + 2t + 1) = 0 \Leftrightarrow t = 1.$$

Returning to the notations it follows $e^{\frac{x}{3}} = 1 \Leftrightarrow x = 0$

Reciprocal $x = 0$ verify the equation.

We deduce that $x = 0$ is the unique solution of the equation.

SP.538 In acute ΔABC the following relationship holds:

$$36 \leq 4\left(\sum_{cyc} \tan A \tan B\right) \leq 9 + \prod_{cyc} \tan^2 A$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$\frac{1}{3}\sum_{cyc}\tan A \overset{AM-HM}{\geq} \frac{3}{\sum_{cyc}\frac{1}{\tan A}}$$

$$\left(\sum_{cyc}\tan A\right)\cdot\left(\sum_{cyc}\frac{1}{\tan A}\right) \geq 9$$

$$\prod_{cyc}\tan A \cdot \frac{\tan A\tan B + \tan B\tan C + \tan C\tan A}{\prod_{cyc}\tan A} \geq 9$$

$$\sum_{cyc}\tan A\tan B \geq 9 \Rightarrow 4\sum_{cyc}\tan A\tan B \geq 36$$

By Padoa's inequality:

$$\prod_{cyc}(\tan A + \tan B - \tan C) \leq \prod_{cyc}\tan A \quad (1)$$

Denote: $u = \sum_{cyc}\tan A = \prod_{cyc}\tan A > 0$

By (1):

$$\prod_{cyc}(u - \tan A) \leq u$$

$$u^3 - 2\left(\sum_{cyc}\tan A\right)\cdot u^2 + 4\left(\sum_{cyc}\tan A\tan B\right)\cdot u - 8u \leq u$$

$$u^3 - 2u^3 + 4\left(\sum_{cyc}\tan A\tan B\right)\cdot u \leq 9u$$

$$4\left(\sum_{cyc}\tan A\tan B\right)\cdot u \leq 9u + u^3$$

$$4\sum_{cyc}\tan A\tan B \leq u^2 + 9$$

$$4\sum_{cyc}\tan A\tan B \leq 9 + \prod_{cyc}\tan^2 A$$

Equality holds for:

$$\tan A = \tan B = \tan C \Rightarrow A = B = C = \frac{\pi}{3}$$

Solution 2 by Marin Chirciu – Romania

Changing the variable $A \to \frac{\pi}{2} - A$, the inequality from enunciation is equivalent with:

$$36 \le 4\sum \cot\frac{A}{2}\cot\frac{B}{2} \le 9 + \prod \cot^2\frac{A}{2}$$

Using the inequalities in triangle $\sum \cot\frac{A}{2}\cot\frac{B}{2} = \frac{4R+r}{r}$ and $\prod \cot\frac{A}{2} = \frac{s}{r}$

First inequality.

$$36 \le 4\sum \cot\frac{A}{2}\cot\frac{B}{2} \Leftrightarrow \sum \cot\frac{A}{2}\cot\frac{B}{2} \ge 9 \Leftrightarrow \frac{4R+r}{r} \ge 9 \Leftrightarrow R \ge 2r, \text{(Euler)}$$

Equality holds if and only if the triangle is equilateral.

Second inequality.

$$4\sum \cot\frac{A}{2}\cot\frac{B}{2} \le 9 + \prod \cot^2\frac{A}{2} \Leftrightarrow 4\cdot\frac{4R+r}{r} \ge 9 + \frac{s^2}{r^2} \Leftrightarrow s^2 \ge 16Rr - 5r^2, \text{(Gerretsen)}$$

Equality holds if and only if the triangle is equilateral.

SP.539 If $a > 0;\ f:\mathbb{R} \to \mathbb{R}$ is a function such that:

$$f\left(ax - \frac{1}{a}\right) \le ax \le f(x) - 1;\ (\forall)x \in \mathbb{R} \quad \text{then:}$$

$$f(2) + f(4) + f(8) > \frac{12\sqrt{a}}{a}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$ax \le f(x) - 1 \Rightarrow f(x) \ge \frac{ax+1}{a} \quad (1)$$

$$f\left(ax - \frac{1}{a}\right) \le ax \quad (2)$$

Replace $y = ax - \frac{1}{a}$ in (2): $y + \frac{1}{a} = ax \Rightarrow x = \frac{1}{a}y + \frac{1}{a^2}$

$$f(y) \le a\left(\frac{1}{a}y + \frac{1}{a^2}\right) \Rightarrow f(y) \le y + \frac{1}{a} = \frac{ay+1}{a}$$

$$f(x) \le \frac{ax+1}{a} \quad (3)$$

By (2);(3):

$$f(x)=\frac{ax+1}{a}\Rightarrow f(x)=x+\frac{1}{a}$$

$$f(2)+f(4)+f(8)=\left(2+\frac{1}{a}\right)+\left(4+\frac{1}{a}\right)+\left(8+\frac{1}{a}\right)\ge$$

$$\overset{AM-GM}{>}6\cdot\sqrt[6]{2\cdot\frac{1}{a}\cdot4\cdot\frac{1}{a}\cdot8\cdot\frac{1}{a}}=6\sqrt[6]{2^6\cdot\frac{1}{a^3}}=12\cdot\sqrt[6]{\frac{1}{a^3}}=12\cdot\frac{1}{\sqrt{a}}=\frac{12\sqrt{a}}{a}$$

Solution 2 by Marin Chirciu – Romania

We have $f(x)\ge ax+1,(\forall)x\in\mathbb{R}$.

We obtain $f(2)+f(4)+f(8)\ge(2a+1)+(4a+1)+(8a+1)=14a+3>\frac{12\sqrt{a}}{a}$

which follows from $14a+3>\frac{12\sqrt{a}}{a},a>0$

We denote $\sqrt{a}=t>0$ and $14a+3>\frac{12\sqrt{a}}{a}$ we write $14t^2+3>\frac{12}{t}\Leftrightarrow$

$\Leftrightarrow 14t^3+3t+12>0,t>0$

We consider the function $g(t)=14t^3+3t-12,t>0$

$g'(t)=42t^2+3>0\Rightarrow g$ is strictly increasing on $(0,\infty)$.

From the table of variation it follows $g(t)>0$ for $t\ge1$.

SP.540 If $x,y\in[3,4];z,t\in[1,12]$ then:

$$(x+y+z+t)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right)\le\frac{100}{3}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$x\in[3,4]\Rightarrow3\le x\le4\Rightarrow x-3\ge0;x-4\le0$$

$$\Rightarrow(x-3)(x-4)\le0\Rightarrow x^2-7x+12\le0$$

$$\Rightarrow x^2+12\le7x\Rightarrow x+\frac{12}{x}\le7\Rightarrow7\ge x+\frac{12}{x}\quad(1)$$

Analogous: $y\in[3,5]\Rightarrow7\ge y+\frac{12}{y}\quad(2)$

$$z\in[1,12]\Rightarrow1\le z\le12\Rightarrow z-1\ge0;z-12\le0$$

$$\Rightarrow(z-1)(z-12)\le0\Rightarrow z^2-13z+12\le0$$

$$\Rightarrow z^2+12\le13z\Rightarrow z+\frac{12}{z}\le13\Rightarrow13\ge z+\frac{12}{z}\quad(3)$$

Analogous: $t\in[1,12]\Rightarrow13\ge t+\frac{12}{t}\quad(4)$

By adding (1);(2);(3);(4):

$$40\geq x+y+z+t+12\left(\frac{1}{x}+\frac{1}{t}+\frac{1}{z}+\frac{1}{t}\right)\geq$$

$$\overset{AM-GM}{\geq}2\sqrt{(x+y+z+t)\cdot 12\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right)}=$$

$$=4\sqrt{3(x+y+z+t)\cdot 3\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right)}$$

$$10\geq\sqrt{3(x+y+z+t)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right)}$$

$$100\geq 3(x+y+z+t)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right)$$

$$(x+y+z+t)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right)\leq\frac{100}{3}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM − GM inequality, we have

$$(x+y+z+t)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right)\leq\left(\frac{x+y+z+t}{4\sqrt{3}}+\sqrt{3}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right)\right)^2$$

$$=\left[\sum_{x,y,z,t}\left(\frac{x}{4\sqrt{3}}+\frac{\sqrt{3}}{x}\right)\right]^2.$$

We have $(x-3)(x-4)\leq 0$, **then** $x^2+12\leq 7x$ **or** $\frac{x}{4\sqrt{3}}+\frac{\sqrt{3}}{x}\leq\frac{7}{4\sqrt{3}}$.

Similarly, we have $\frac{y}{4\sqrt{3}}+\frac{\sqrt{3}}{y}\leq\frac{7}{4\sqrt{3}}$.

Also, we have $(z-1)(z-12)\leq 0$, **then** $z^2+12\leq 13z$ **or** $\frac{z}{4\sqrt{3}}+\frac{\sqrt{3}}{z}\leq\frac{13}{4\sqrt{3}}$.

Similarly, we obtain $\frac{t}{4\sqrt{3}}+\frac{\sqrt{3}}{t}\leq\frac{13}{4\sqrt{3}}$.

Therefore

$$(x+y+z+t)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right)\leq\left(2.\frac{7}{4\sqrt{3}}+2.\frac{13}{4\sqrt{3}}\right)^2=\frac{100}{3}.$$

Equality holds iff $x,y\in\{3,4\}$ **and** $z,t\in\{1,12\}$.

UNDERGRADUATE PROBLEMS

UP.526 Prove the identity:

$$\int_0^\infty \frac{|\sin(x)|}{1+x^2}dx = 1 - 2\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)e^{2n}}$$

Proposed by Vasile Mircea Popa – Romania

Solution 1 by proposer

Let us denote:

$$I = \int_0^\infty \frac{|\sin(x)|}{1+x^2}dx$$

The function $f(x) = |\sin(x)|$ is periodic with period π and satisfies Dirichlet's conditions. Also, the function is even. We expand the function in the Fourier series:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2nx)$$

where:

$$a_0 = \frac{1}{\pi}\int_0^\pi \sin(x)\,dx;\ a_n = \frac{2}{\pi}\int_0^\pi \sin(x)\cos(2nx)\,dx$$

Calculating these integrals, we obtain:

$$a_0 = \frac{2}{\pi};\ a_n = -\frac{4}{\pi}\cdot\frac{1}{4n^2-1}$$

We have:

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2nx)}{4n^2-1}$$

Substituting $f(x)$ in the expression of I, result:

$$I = \frac{2}{\pi}\int_0^\infty \frac{1}{1+x^2}dx - \frac{4}{\pi}\int_0^\infty \sum_{n=1}^{\infty}\frac{\cos(2nx)}{(4n^2-1)(1+x^2)}$$

So:

$$I=1-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\int_0^{\infty}\frac{\cos(2nx)}{1+x^2}dx$$

We now use the following relationship:

$$\int_0^{\infty}\frac{\cos(mx)}{1+x^2}dx=\frac{\pi}{2}e^{-m}, m>0$$

This relation is Laplace's integral and is well known. It is easily proved for example using the properties of the Laplace transform. We obtained the value of the integral I:

$$I=1-2\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)e^{2n}}$$

Solution 2 by Pham Duc Nam-Vietnam

*** The function $f(x)=|\sin(x)|$ is even function then $b_n=0$ and the Fourier series collapses to**

$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos(nx)$ **and** $a_0=\frac{2}{\pi}\int_0^{\pi}f(x)\,dx, a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos(nx)\,dx$

$$x\in[0,\pi]\Rightarrow\sin(x)\ge 0\Rightarrow|\sin(x)|=\sin(x)$$

Then the Fourier series of $|\sin(x)|$ is given by:

$$f(x)=\frac{1}{\pi}\int_0^{\pi}\sin(x)\,dx+\sum_{n=2}^{\infty}\left(\frac{2}{\pi}\int_0^{\pi}\sin(x)\cos(nx)\,dx\right)\cos(nx)\because a_1=0$$

1. $\frac{1}{\pi}\int_0^{\pi}\sin(x)\,dx=\frac{1}{\pi}(-\cos(x))|_0^{\pi}=\frac{2}{\pi}$

2. $\frac{2}{\pi}\int_0^{\pi}\sin(x)\cos(nx)\,dx=\frac{1}{\pi}\int_0^{\pi}\left(\sin(x(1+n))+\sin\left(x(1-n)\right)\right)dx=$

$$=-\frac{1}{\pi}\frac{2(\cos(\pi n)+1)}{n^2-1}=\begin{cases}0, if\ n\ is\ odd\\ -\frac{1}{\pi}\frac{4}{n^2-1}, if\ n\ is\ even\end{cases}\Rightarrow f(x)=\frac{2}{\pi}-\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2kx)}{4k^2-1}$$

$$*\int_0^{\infty}\frac{|\sin(x)|}{1+x^2}dx=\int_0^{\infty}\frac{1}{1+x^2}\left(\frac{2}{\pi}-\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2kx)}{4k^2-1}\right)dx=$$

$$=\frac{2}{\pi}\int_0^{\infty}\frac{1}{1+x^2}dx-\frac{4}{\pi}\sum_{k}^{\infty}\frac{1}{4k^2-1}\int_0^{\infty}\frac{\cos(2kx)}{1+x^2}dx$$

$$=1-\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\left(\frac{1}{2}\mathcal{R}\int_{-\infty}^{\infty}\frac{e^{2kix}}{1+x^2}dx\right)=$$

$$= 1 - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\left(\frac{1}{2}\mathcal{R}\left(2\pi i.\,Res\left(\frac{e^{2kix}}{1+z^2}, z = i\right)\right)\right)$$

$$= 1 - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\left(\frac{1}{2}\mathcal{R}\left(2\pi i \lim_{x\to i}(z-i)\frac{e^{2kix}}{(z+i)(z-i)}\right)\right) =$$

$$= 1 - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\left(\frac{1}{2}\mathcal{R}\left(2\pi i\left(-\frac{1}{2}ie^{-2k}\right)\right)\right) = 1 - 2\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\frac{1}{e^{2k}},$$ as required.

Solution 3 by Ankush Kumar Parcha-India

Consider, $f(z) \coloneqq \frac{e^{iz}}{1+z^2}\coth\left(\frac{\pi z}{2}\right) \& \Gamma: \gamma \cup (-r,-\delta) \cup \psi \cup (\delta, R)$

is the contour containing γ and ψ a semi-circular arcs in the upper half of $\mathbb{C}$ - plane with different radii and $(-R,-\delta) \cup (\delta, R)$ lies on $\mathcal{R}(z)$ axis

$$\oint_{\Gamma}\frac{e^{iz}}{1+z^2}\coth\left(\frac{\pi z}{2}\right)dz = \int_{\gamma}\frac{e^{iz}}{1+z^2}\coth\left(\frac{\pi z}{2}\right)dz + \int_{-R}^{-\delta}\frac{e^{ix}}{1+x^2}\coth\left(\frac{\pi x}{2}\right)dx$$

$$+\int_{\psi}\frac{e^{iz}}{1+z^2}\coth\left(\frac{\pi z}{2}\right)dz + \int_{\delta}^{R}\frac{e^{ix}}{1+x^2}\coth\left(\frac{\pi x}{2}\right)dx$$

($\because$ Cauchy's Residue Theorem: $\int_C f(z)\,dz = 2\pi i\sum_k Res f(z = z_k)$)

($\because$ Jordan's Lemma: $\lim\limits_{R\to\infty}\int_{C_R} e^{imz} f(z)\,dz = 0, f(z) \to 0 \wedge z \to \infty \wedge m \in \mathbb{R}^+$)

$$\overset{Taking\ limits}{\underset{R\to\infty,\delta\to 0}{\Longrightarrow}} \lim_{\substack{R\to\infty\\ \delta\to 0}} 2\pi i\, Res\, f(z=i) + \lim_{\substack{R\to\infty\\ \delta\to 0}} 2\pi i\sum_{n\in\mathbb{N}} Res\, f(z = 2ni) =$$

$$= \underbrace{\lim_{\substack{R\to\infty\\ \delta\to 0}}\int_{\gamma}\frac{e^{iz}}{1+z^2}\coth\left(\frac{\pi z}{2}\right)dz}_{=0, \because f(z)\to 0 \wedge z\to\infty\wedge 1\in\mathbb{R}^+}$$

$$+\lim_{\substack{R\to\infty\\ \delta\to 0}}\int_{-R}^{-\delta}\frac{e^{ix}}{1+x^2}\coth\left(\frac{\pi x}{2}\right)dx + \underbrace{\lim_{\substack{R\to\infty\\ \delta\to 0}}\int_{\psi}\frac{e^{iz}}{1+z^2}\coth\left(\frac{\pi z}{2}\right)dz}_{=-i\pi\, Res\, f(z=0)} +$$

$$+\lim_{\substack{R\to\infty\\ \delta\to 0}}\int_{\delta}^{R}\frac{e^{ix}}{1+x^2}\coth\left(\frac{\pi x}{2}\right)dx$$

$$\Rightarrow 2\pi i \underbrace{\lim_{z\to i}\frac{(z-i)e^{iz}\coth\left(\frac{\pi z}{2}\right)}{1+z^2}}_{=0}+2\pi i\sum_{n\in\mathbb{N}}\lim_{z\to 2ni}\frac{(z-2ni)e^{iz}}{1+z^2}\coth\left(\frac{\pi z}{2}\right)$$

$$=\int_{\mathbb{R}^-}\frac{e^{ix}}{1+x^2}\coth\left(\frac{\pi x}{2}\right)dx-i\pi\underbrace{\lim_{z\to 0}\frac{ze^{iz}}{1+z^2}\coth\left(\frac{\pi z}{2}\right)}_{=\frac{2}{\pi}}+\int_{\mathbb{R}^+}\frac{e^{ix}}{1+x^2}\coth\left(\frac{\pi x}{2}\right)dx$$

$$\overset{Taking\ Imaginary}{\underset{part}{\Longrightarrow}}\ \mathfrak{I}\left\{2\pi i\left(-\frac{2}{3\pi e^2}-\frac{2}{15\pi e^4}-\frac{2}{35\pi e^6}-\cdots\right)\right\}=\mathfrak{I}\{-2i\}$$

$$+\mathfrak{I}\int_{\mathbb{R}}\frac{e^{ix}}{1+x^2}\coth\left(\frac{\pi x}{2}\right)dx\Rightarrow 2-4\sum_{n\in\mathbb{N}}\frac{1}{(4n^2-1)e^{2n}}=\int_{\mathbb{R}}\frac{\sin(x)}{1+x^2}\coth\left(\frac{\pi x}{2}\right)dx$$

$$\left(\because\int_{-a}^{+a}f(t)\,dt=2\int_0^a f(t),f(-t)=f(t)\right)$$

$$\left(\because \textit{Maz Identity}:\int_{\mathbb{R}^+}f(t)\cdot g(t)dt=\int_{\mathbb{R}^+}\mathcal{L}_t\{f(t)\}(x)\cdot\mathcal{L}_t^{-1}\{g(t)\}(x)dx\right)$$

$$\Rightarrow 1-2\sum_{n\in\mathbb{N}}\frac{1}{(4n^2-1)e^{2n}}=\int_{R^+}\mathcal{L}_x\{\sin(x)\}(t)\cdot\mathcal{L}_x^{-1}\left\{\frac{\coth\left(\frac{\pi x}{2}\right)}{1+x^2}\right\}(t)dt$$

$$\left(\because\mathcal{L}_t\{|\sin(t)|\}(s)=\frac{\coth\left(\frac{\pi s}{2}\right)}{1+s^2}\right)$$

$$\left(\because\mathcal{L}_t\left\{\sin(\omega t)\,(s)=\frac{\omega}{s^2+\omega^2}\right\},s>|\mathfrak{I}(\omega)|\right)\overset{t-x}{\Longrightarrow}\int_{\mathbb{R}^+}\frac{|\sin(x)|}{1+x^2}dx$$

$$=1-2\sum_{n\in\mathbb{N}}\frac{1}{(4n^2-1)e^{2n}}$$

UP.527 Prove the closed form:

$$\int_0^\infty\frac{\ln x}{x^3+x\sqrt{x}+1}dx=-\frac{32\pi^2}{81}\sin\frac{\pi}{18}$$

Proposed by Vasile Mircea Popa – Romania

Solution 1 by proposer

Let us denote:

$$I=\int_0^\infty \frac{\ln x}{x^3+x\sqrt{x}+1}dx;A=\int_0^1 \frac{\ln x}{x^3+x\sqrt{x}+1}dx;B=\int_1^\infty \frac{\ln x}{x^3+x\sqrt{x}+1}dx$$

We consider the integral A. We make the variable change: $x=y^{\frac{2}{3}}$

We have, successively:

$$A=\frac{4}{9}\int_0^1 \frac{(1-y)y^{-\frac{1}{3}}\ln y}{1-y^3}dy=\frac{4}{9}\left(\int_0^1 \frac{y^{\frac{1}{3}}\ln y}{1-y^3}dy-\int_0^1 \frac{y^{\frac{2}{3}}\ln y}{1-y^3}dy\right)$$

$$A=\frac{4}{9}\left(\int_0^1 \sum_{k=0}^{\infty} y^{3k-\frac{1}{3}}\ln y\,dy-\int_0^1 \sum_{k=0}^{\infty} y^{3k-\frac{2}{3}}\ln y\,dy\right);$$

$$A=\frac{4}{9}\sum_{k=0}^{\infty}\left(\int_0^1 y^{3k-\frac{1}{3}}\ln y\,dy-\int_0^1 y^{3k+\frac{2}{3}}\ln y\,dy\right)$$

We will use the following relationship:

$$\int_0^1 x^a \ln x\,dx=-\frac{1}{(a+1)^2}, \text{ where } a\in\mathbb{R}, a\ge 0.$$

We obtain:

$$A=\frac{4}{9}\sum_{k=0}^{\infty}\left[\frac{1}{\left(3k+\frac{5}{3}\right)^2}-\frac{1}{\left(3k+\frac{2}{3}\right)^2}\right];\ A=\frac{4}{9}\sum_{k=0}^{\infty}\left[\frac{\frac{1}{9}}{\left(k+\frac{5}{9}\right)^2}-\frac{\frac{1}{9}}{\left(k+\frac{2}{9}\right)^2}\right]$$

We now use the following relationship:

$$\Psi_1(x)=\sum_{n=0}^{\infty}\frac{1}{(x+n)^2}$$

where $\Psi_1(x)$ is the trigamma function.We obtained the value of the integral A:

$$A=\frac{4}{81}\left[\Psi_1\left(\frac{5}{9}\right)-\Psi_1\left(\frac{2}{9}\right)\right]$$

We consider the integral B. We make the variable change: $x=\frac{1}{y}$. Then, by proceeding to the integral A, we obtain:

$$B=\frac{4}{81}\left[\Psi_1\left(\frac{4}{9}\right)-\Psi_1\left(\frac{7}{9}\right)\right]$$

Result:

$$I=A+B=\frac{4}{81}\left[\Psi_1\left(\frac{5}{9}\right)-\Psi_1\left(\frac{2}{9}\right)+\Psi_1\left(\frac{4}{9}\right)-\Psi_1\left(\frac{7}{9}\right)\right]$$

We use the reflection formula:

$$\Psi_1(x)+\Psi_1(1-x)=\frac{\pi^2}{\sin^2(\pi x)}$$

We obtain:

$$\Psi_1\left(\frac{2}{9}\right)+\Psi_1\left(\frac{7}{9}\right)=\frac{\pi^2}{\sin^2\frac{2\pi}{9}};\ \Psi_1\left(\frac{4}{9}\right)+\Psi_1\left(\frac{5}{9}\right)=\frac{\pi^2}{\sin^2\frac{4\pi}{9}}$$

$$I=\frac{4\pi^2}{81}\left(-\frac{1}{\sin^2\frac{2\pi}{9}}+\frac{1}{\sin^2\frac{4\pi}{9}}\right)$$

We have:

$$-\frac{1}{\sin^2\frac{2\pi}{9}}+\frac{1}{\sin^2\frac{4\pi}{9}}=-8\sin\frac{\pi}{18}$$

We will prove this equality. We use the relationships

$$\sin 3a=\sin a\,(1+2\cos 2a);\cos 3a=\cos a\,(2\cos 2a-1)$$

We consider

$$E=\frac{1}{\sin^2\frac{2\pi}{9}}-\frac{1}{\sin^2\frac{4\pi}{9}}=\frac{1}{\sin^2\frac{2\pi}{9}}-\frac{1}{\cos^2\frac{\pi}{18}}=\frac{\left(1+2\cos\frac{4\pi}{9}\right)^2}{\sin^2\frac{2\pi}{3}}-\frac{\left(2\cos\frac{\pi}{9}-1\right)^2}{\cos^2\frac{\pi}{6}}$$

$$E=\frac{16}{3}\left(\cos\frac{4\pi}{9}+\cos\frac{\pi}{9}\right)\left(1+\cos\frac{4\pi}{9}-\cos\frac{\pi}{9}\right)=\frac{16\sqrt{3}}{3}\cos\frac{5\pi}{18}\left(1-\sin\frac{5\pi}{18}\right)$$

$$E=\frac{8\sqrt{3}}{3}\left(2\cos\frac{5\pi}{18}-\sin\frac{5\pi}{9}\right)$$

But: $\cos\frac{5\pi}{18}=\frac{1}{2}\cos\frac{\pi}{18}+\frac{\sqrt{3}}{2}\sin\frac{\pi}{18}$; $\sin\frac{5\pi}{9}=\cos\frac{\pi}{18}$. **So:** $E=8\sin\frac{\pi}{18}$

Result:

$$I=-\frac{32\pi^2}{81}\sin\frac{\pi}{18}$$

Solution 2 by Odeyemi Gideon-Nigeria

$$\int_0^{+\infty}\frac{\ln(x)}{x^3+x\sqrt{x}+1}dx=\int_0^{+\infty}\frac{\left(x^{\frac{3}{2}}-1\right)\ln(x)}{x^{\frac{9}{2}}-1}dx=\int_0^{+\infty}\left(\frac{x^{\frac{3}{2}}\ln(x)}{x^{\frac{9}{2}}-1}-\frac{\ln(x)}{x^{\frac{9}{2}}-1}\right)dx$$

$$\overset{x^{\frac{9}{2}}=x}{=}\frac{4}{81}\int_0^{+\infty}\left(\frac{x^{-\frac{4}{9}}\ln(x)}{x-1}-\frac{x^{-\frac{7}{9}}\ln(x)}{x-1}\right)dx$$

Let's consider $\int_0^{+\infty}\frac{x^a\ln(x)}{x-1}dx$

$$\int_0^{+\infty}\frac{x^a\ln(x)}{x-1}dx=\frac{d}{da}\left(\int_0^{+\infty}\frac{x^a}{x-1}dx\right)\overset{\frac{1}{x-1}=x}{=}\frac{d}{da}\left((-1)^a\int_0^1 x^{-a-1}(1-x)^a dx\right)$$

$$=\frac{d}{da}\left((-1)^a\Gamma(-a)\Gamma(a+1)\right)\overset{E.R.F}{=}\frac{d}{da}\left((-1)^a\frac{\pi}{\sin(-\pi a)}\right)=\frac{d}{da}\left((-1)^{a+1}\frac{\pi}{\sin(\pi a)}\right)$$

$$\Rightarrow\frac{4}{81}\int_0^{+\infty}\left(\frac{x^{-\frac{4}{9}}\ln(x)}{x-1}-\frac{x^{-\frac{7}{9}}\ln(x)}{x-1}\right)dx=\frac{4}{81}\left(\pi^2\csc^2(\pi a)\right)_{a=-\frac{7}{9}}^{x=-\frac{4}{9}}$$

$$=\frac{4\pi^2}{81}\left(\sec^2\left(\frac{\pi}{18}\right)-\csc^2\left(\frac{2\pi}{9}\right)\right)=-\frac{32\pi^2}{81}\sin\left(\frac{\pi}{18}\right)$$

Solution 3 by Pham Duc Nam-Vietnam

$$\Omega=\int_0^{\infty}\frac{\ln(x)}{x^3+x\sqrt{x}+1}dx=-\frac{32}{81}\pi^2\sin\left(\frac{\pi}{18}\right)?$$

$$\Omega=\int_0^{\infty}\frac{\ln(x)}{x^3+x\sqrt{x}+1}dx, t=x\sqrt{x}\Rightarrow dx=\frac{2}{3}\frac{1}{t^{\frac{1}{3}}}dt$$

$$\Rightarrow\Omega=\frac{4}{9}\int_0^{\infty}\frac{t^{-\frac{1}{3}}\ln(t)}{t^2+t+1}dt=\frac{4}{9}\frac{\partial}{\partial s}\bigg|_{s=\frac{2}{3},x=\frac{1}{2}}\int_0^{\infty}\frac{t^{s-1}}{t^2+2xt+1}dt=$$

$$=\frac{4}{9}\frac{\partial}{\partial s}\bigg|_{s=\frac{2}{3},x=\frac{1}{2}}\int_0^{\infty}t^{s-1}\left(\sum_{n=0}^{\infty}U_n(x)(-t)^n\right)dt$$

Where: $U_n(x)$ is the Chebyshev polynomials of the second kind, and its generating function is:

$$\sum_{n=0}^{\infty}U_n(x)\,t^n=\frac{1}{t^2-2xt+1},\text{ letting } x=\cos(\theta)\Rightarrow U_n(x)=\frac{\sin((n+1)\theta)}{\sin(\theta)}$$

$$\Rightarrow\Omega=\frac{4}{9}\frac{\partial}{\partial s}\bigg|_{s=\frac{2}{3},\theta=\frac{\pi}{3}}\int_0^{\infty}t^{s-1}\left(\sum_{n=0}^{\infty}\frac{\Gamma(n+1)\sin((n+1)\theta)}{\sin(\theta)n!}(-t)^n\right)dt$$

by Ramanujan's master theorem:

$$f(x)=\sum_{n=0}^{\infty}\frac{\varphi(n)}{n!}(-x)^n\Rightarrow\mathcal{M}\{f(x)\}(s)=\Gamma(s)\varphi(-s),$$

apply for Ω with $\varphi(n)=\Gamma(n+1)\sin\big((n+1)\theta\big)$

$$\Rightarrow \Omega=\frac{4}{9}\frac{\partial}{\partial s}\bigg|_{s=\frac{2}{3},\theta=\frac{\pi}{3}}\Gamma(s)\Gamma(1-s)\frac{\sin\big((1-s)\theta\big)}{\sin(\theta)}=\frac{8\pi}{9\sqrt{3}}\frac{\partial}{\partial s}\bigg|_{s=\frac{2}{3}}\frac{\sin\left((1-s)\frac{\pi}{3}\right)}{\sin(\pi s)}$$

$$=\frac{8\pi}{9\sqrt{3}}\left(-\frac{\pi}{3\sin(\pi s)}\left(\cos\left(\frac{\pi}{3}(s-1)\right)+3\cos\left(\frac{\pi}{6}(2s+1)\right)\cot(\pi s)\right)\right)_{s=\frac{2}{3}}=$$

$$=-\frac{16\pi^2}{81}\left(\cos\left(\frac{\pi}{9}\right)-\sqrt{3}\sin\left(\frac{\pi}{9}\right)\right)$$

$$=-\frac{32\pi^2}{81}\left(\frac{1}{2}\cos\left(\frac{\pi}{9}\right)-\frac{\sqrt{3}}{2}\sin\left(\frac{\pi}{9}\right)\right)=-\frac{32\pi^2}{81}\left(\sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{9}\right)-\cos\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{9}\right)\right)$$

$$=-\frac{32\pi^2}{81}\sin\left(\frac{\pi}{6}-\frac{\pi}{9}\right)=-\frac{32\pi^2}{81}\sin\left(\frac{\pi}{81}\right),$$ hence proved.

Solution 4 by Ankush Kumar Parcha-India

We have,

$$\int_{\mathbb{R}^+}\frac{\ln(x)}{x^3+x\sqrt{x}+1}dx\overset{x\to x^2}{\Rightarrow}4\int_{\mathbb{R}^+}\frac{x\ln(x)}{x^6+x^3+1}dx$$

$$\Rightarrow 4\int_0^1\frac{(x-x^4)\ln(x)}{1-x^9}dx+4\underbrace{\int_1^\infty\frac{(x-x^4)\ln(x)}{1-x^9}dx}_{x\to\frac{1}{x}}$$

$$\Rightarrow 4\int_0^1\frac{(x^6-x^4-x^3+x)\ln(x)}{1-x^9}dx\overset{x^9\to x}{\Rightarrow}\frac{4}{81}\int_0^1\frac{\left(x^{-\frac{7}{9}}-x^{-\frac{5}{9}}-x^{-\frac{4}{9}}+x^{-\frac{2}{9}}\right)\ln(x)}{1-x}dx$$

$$\left(\because \psi^{(m)}(z)=-\int_0^1\frac{t^{z-1}\ln^m(t)}{1-t}dt,\mathcal{R}(z)>0\wedge m>0\right)$$

$$\Rightarrow\frac{4}{81}\left[\psi^{(1)}\left(\frac{5}{9}\right)+\psi^{(1)}\left(\frac{4}{9}\right)-\psi^{(1)}\left(\frac{7}{9}\right)-\psi^{(1)}\left(\frac{2}{9}\right)\right]$$

$$\left(\because \psi^{(1)}(1-z)+\psi^{(1)}(z)=\frac{\pi^2}{\sin^2(\pi z)}\right)$$

$$\Rightarrow\frac{4}{81}\left(\frac{\pi^2}{\sin^2\left(\frac{4\pi}{9}\right)}-\frac{\pi^2}{\sin^2\left(\frac{2\pi}{9}\right)}\right)\overset{\because\cos(2x)=1-2\sin^2(x)}{\Rightarrow}\frac{4\pi^2}{81}\left(\frac{1}{\cos^2\left(\frac{\pi}{18}\right)}-\frac{2}{1-\sin\left(\frac{\pi}{18}\right)}\right)$$

$$\Rightarrow-\frac{4\pi^2}{81}\left(\frac{1+2\sin\left(\frac{\pi}{18}\right)}{\cos^2\left(\frac{\pi}{18}\right)}\right)\overset{\because\sin(3x)=3\sin(x)-4\sin^3(x)}{\Rightarrow}$$

$$\Rightarrow \int_{\mathbb{R}^+} \frac{\ln(x)}{x^3 + x\sqrt{x} + 1} dx = -\frac{32\pi^2}{81} \sin\left(\frac{\pi}{18}\right)$$

Solution 5 by Djamel Arrouche-Algeria

$$\int_0^\infty \frac{\ln(x)}{x^3 + x\sqrt{x} + 1} dx = \Delta = -\frac{32\pi^2}{81} \sin\left(\frac{\pi}{18}\right)$$

$$x\sqrt{x} = y; dx = \frac{2}{3} y^{-\frac{1}{3}} \Rightarrow$$

$$\Delta = \frac{4}{9} \int_0^\infty \frac{\ln(y)\, y^{-\frac{1}{3}} dy}{y^2 + y + 1} = \frac{4}{9}\left(\int_0^1 \frac{y^{-\frac{1}{3}} \ln(y)}{y^2 + y + 1} + \int_1^\infty \frac{y^{-\frac{1}{3}} \ln(y)}{y^2 + y + 1} dy_{y \to \frac{1}{z}}\right)$$

$$= \frac{4}{9} \int_0^1 \frac{y^{-\frac{1}{3}} \ln(y)}{y^2 + y + 1} - \frac{4}{9} \int_0^1 \frac{z^{\frac{1}{3}} \ln(z)\, dz}{z^2 + z + 1} = \frac{4}{9} \int_0^1 \frac{y^{\frac{1}{3}} - y^{-\frac{1}{3}}}{y^2 + y + 1} \ln(y)\, dy$$

$$= \frac{4}{9}\left(f\left(-\frac{1}{3}\right) + f\left(-\frac{1}{3}\right)\right) \Bigg| f(s) = \int_0^1 \frac{y^s \ln(y)}{y^2 + y + 1} = \int_0^1 \frac{y^s - y^{s+1}}{1 - y^3} \ln(y)$$

$$y^3 = z \Rightarrow f(s) = \frac{1}{9} \int_0^1 \frac{z^{\frac{s-2}{3}} \ln(z)}{1 - z} - \frac{1}{9} \int_0^1 \frac{z^{\frac{s-1}{3}} \ln(z)}{1 - z}$$

we have $\Psi^{(1)}(1 + z) = \int_0^1 \frac{-\ln(x) x^z}{1-x} dx$

$$\Rightarrow f(s) = \frac{1}{9}\left(\Psi^{(1)}\left(\frac{s + 2}{3}\right) - \Psi^{(1)}\left(\frac{s + 1}{3}\right)\right)$$

$$\Delta = \frac{4}{9}\left(f\left(-\frac{1}{3}\right) + f\left(\frac{1}{3}\right)\right) = \frac{4}{9} \cdot \frac{1}{9}\left(-\Psi^1\left(\frac{7}{9}\right) + \Psi^1\left(\frac{4}{9}\right) - 1 + \Psi^1\left(\frac{5}{9}\right) - \Psi^1\left(\frac{2}{9}\right)\right)$$

$$= \frac{4}{81}\left(-\left\{\Psi^1\left(\frac{14}{18}\right) + \Psi^1\left(\frac{4}{18}\right)\right\} + \left(\Psi^1\left(\frac{8}{18}\right) + \Psi^1\left(\frac{10}{18}\right)\right)\right)$$

$$\Psi^1(1 - z) + \Psi^1(z) = \frac{\pi^2}{\sin^2(\pi z)}$$

$$\Delta = \frac{4\pi^2}{81}\left(-\frac{1}{\sin^2\left(\frac{4\pi}{18}\right)} + \frac{1}{\sin^2\left(\frac{8\pi}{18}\right)}\right)$$

$$\frac{1}{\sin^2\left(\frac{8\pi}{18}\right)} - \frac{1}{\sin^2\left(\frac{4\pi}{18}\right)} = -8 \sin\left(\frac{\pi}{18}\right) ..?$$

$$\Leftrightarrow \frac{1 - 4\cos^2\left(\frac{4\pi}{18}\right)}{\sin^2\left(\frac{8\pi}{18}\right)} = -8 \sin\left(\frac{\pi}{18}\right)$$

$$\Leftrightarrow = \frac{1-2\left(\cos\left(\frac{8\pi}{18}\right)+1\right)}{\cos^2\left(\frac{\pi}{18}\right)} = \frac{-1-2\sin\left(\frac{\pi}{18}\right)}{\cos^2\left(\frac{\pi}{18}\right)} = -8\sin\left(\frac{\pi}{18}\right)$$

$$\Leftrightarrow -1-2\sin\left(\frac{\pi}{18}\right) = 8\sin\frac{\pi}{18}\cos^2\frac{\pi}{18} = -4\sin\frac{\pi}{18}\left(\cos\left(\frac{2\pi}{18}\right)+1\right)$$

$$\Leftrightarrow -1+2\sin\left(\frac{\pi}{18}\right) = -2\left(2\sin\left(\frac{\pi}{18}\right)\cos\left(\frac{2\pi}{18}\right)\right) = -2\left(\sin\left(\frac{\pi}{18}+\frac{2\pi}{18}\right)\right)+\sin\left(\frac{\pi}{18}-\frac{2\pi}{18}\right)$$

$$\Leftrightarrow -1+2\sin\left(\frac{\pi}{18}\right) = -2\sin\left(\frac{\pi}{6}\right)-2\sin\left(-\frac{\pi}{18}\right) .. \text{ True}$$

$$\Rightarrow \frac{1}{\sin^2\left(\frac{8\pi}{18}\right)} - \frac{1}{\sin^2\left(\frac{4\pi}{18}\right)} = -8\sin\left(\frac{\pi}{18}\right)$$

$$\int_0^\infty \frac{\ln(x)}{x^2+x\sqrt{x}+1} = \frac{4\pi^2}{81}\left(\frac{1}{\sin^2\left(\frac{8\pi}{18}\right)} - \frac{1}{\sin^2\left(\frac{4\pi}{18}\right)}\right) = -\frac{32\pi^2}{81}\sin\left(\frac{\pi}{18}\right)$$

UP.528 If $a_n > 0; r_n > 0; a_{n+1} = a_n + n\cdot r_n; n \in \mathbb{N}^*$ and

$$\lim_{n\to\infty} r_n = r > 0$$

then find:

$$\Omega = \lim_{n\to\infty}(2H_n - \log a_n)$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

Solution 1 by proposers

$$\lim_{n\to\infty}\frac{a_n}{n^2} \overset{CESARO-STOLZ}{=} \lim_{n\to\infty}\frac{a_{n+1}-a_n}{(n+1)^2-n^2} =$$

$$= \lim_{n\to\infty}\frac{a_{n+1}-a_n}{2n+1} = \lim_{n\to\infty}\frac{nr_n}{2n+1} = \lim_{n\to\infty}\frac{n}{2n+1}\cdot r = \frac{r}{2}$$

$$\Omega = \lim_{n\to\infty}(2H_n - \log a_n) = \lim_{n\to\infty}(2H_n - 2\log n + 2\log n - \log a_n) =$$

$$= 2\lim_{n\to\infty}(H_n - \log n) + \lim_{n\to\infty}\log\left(\frac{n^2}{a_n}\right) = 2\gamma + \log\left(\frac{2}{r}\right)$$

Solution 2 by Djamel Arrouche-Algeria

$$a_{n+1} = a_n + n\cdot r_n, \qquad \lim_{n\to\infty} r_n = r$$

$$\Omega = \lim_{n\to\infty}(2H_n - \log(a_n)), \qquad a_{n+1} - a_n = n\cdot r_n$$

$$\frac{a_{n+1}-a_n}{n}=r_n$$

$$\forall\epsilon>0\ \exists N,\forall n\geq N\ r-\epsilon<r_n<r+\epsilon\Rightarrow n(r-\epsilon)<a_{n+1}-a_n<(r+\epsilon)n$$

$$\Rightarrow\sum_{N}^{n}k(r-\epsilon)<\sum_{k=N}^{n}(a_{k+1}-a_k)<\sum_{k=N}^{n}(r+\epsilon)\,k$$

$$\Rightarrow(r-\epsilon)\frac{(n+N)(n-N+1)}{2}<a_{n+1}-a_N<(r+\epsilon)\frac{(n-N+1)(n+N)}{2}$$

$$\Rightarrow\left(1-\frac{\epsilon}{r}\right)\left(1+\frac{n}{N}\right)\left(1+\frac{1-N}{n}\right)<\frac{2}{n^2r}(a_{n+1}-a_N)<$$

$$<\left(1+\frac{\epsilon}{r}\right)\left(1+\frac{n}{N}\right)\left(1+\frac{1-N}{n}\right)$$

ϵ is arbitrary small

$$\Rightarrow\lim_{n\to\infty}\frac{2}{n^2r}(a_{n+1}-a_N)=1$$

Since $a_N\in\mathbb{R}\Rightarrow\lim_{n\to\infty}a_{n+1}\cdot\frac{2}{n^2r}=1$

$$2H_n-\log(a_n)=2H_n+\log\left(\frac{2}{n^2r}\right)-\log\left(\frac{2}{n^2r}a_n\right)=$$

$$=2H_n-2\log(n)+\log(2)+\log\left(\frac{2}{n^2r}a_n\right)$$

$$=2(H_n-\log(n))+\log\left(\frac{2}{r}\right)+\log\left(\frac{2}{n^2r}a_n\right)$$

$\lim\limits_{n\to\infty}\log\left(\frac{a_n\cdot 2}{n^2r}\right)=\log(1)=0;\ \lim\limits_{n\to\infty}H_n-\log(n)=\gamma$: Euler Mascheroni Constant

$$\Omega=\lim_{n\to\infty}(2H_n-\log(a_n)),\qquad \Omega=2\cdot\gamma+\log\left(\frac{2}{r}\right)$$

Solution 3 by Kamel Gandouli Rezgui-Algeria

$$a_{n+1}-a_n=nr_n;n>0$$

$$\Rightarrow\sum_{k=1}^{n-1}a_{k+1}-a_k=\sum_{k=1}^{n-1}kr_k\Rightarrow a_n=\sum_{k=1}^{n-1}kr_k+a_1$$

$$\Rightarrow\log(a_n)=\log\left(\sum_{k=1}^{n-1}kr_k+a_1\right)=\log n^2+\log\left(\frac{1}{n^2}\sum_{k=1}^{n-1}kr_k+\frac{a_1}{n^2}\right)$$

$$= \log n^2 + \log\left(\frac{1}{n^2}\sum_{k=1}^{n} kr_k - \frac{r_n}{n} + \frac{a_1}{n^2}\right)$$

$$\frac{1}{n^2}\sum_{k=1}^{n} k(r_k - r) + \frac{1}{n^2}\sum_{k=1}^{n} kr$$

$\lim\limits_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n} k(r_k - r) = 0$ **Cesaro lemma**$\Rightarrow \lim\limits_{n\to+\infty} \frac{1}{n^2}\sum_{k=1}^{n} k(r_k - r) = 0$

and $\frac{1}{n^2}\sum_{k=1}^{n} kr = \frac{n(n+1)}{2n^2} r \Rightarrow \lim\limits_{n\to+\infty} \frac{1}{n^2}\sum_{k=1}^{n} kr = \frac{r}{2}$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{n^2}\sum_{k=1}^{n} kr_k = \frac{r}{2} \Rightarrow \lim_{n\to+\infty} \log\left(\frac{1}{n^2}\sum_{k=1}^{n} kr_k - \frac{r_n}{n} + \frac{a_1}{n^2}\right) = \log\frac{r}{2}$$

$$\Rightarrow 2H_n - \log a_n = 2H_n - 2\log n + \log\left(\frac{1}{n^2}\sum_{k=1}^{n} kr_k - \frac{r_n}{n} + \frac{a_1}{n^2}\right)$$

$$\Rightarrow \lim_{n\to+\infty} 2H_n - \log a_n = 2\gamma + \log\frac{r}{2}$$

Solution 4 by Le Thu-Vietnam

$$\because a_{n+1} = a_n + nr_n \Rightarrow a_{n+2} - a_{n+1} = (n+1)r_{n+1}$$

$$\because \lim r_{n+1} = \lim r_n = r \in \mathbb{R} \Rightarrow \lim \frac{r_n}{n} = 0$$

$$\therefore \Omega \equiv \lim[2\mathcal{H}_n - \ln(a_n)]$$

$$= \lim[2\mathcal{H}_n - \ln(a_{n+1} - nr_n)]$$

$$= 2\overbrace{\lim[\mathcal{H}_n - \ln(n)]}^{\equiv\gamma} - \lim \ln\left\{\frac{a_{n+1}}{n^2} - \frac{r_n}{n}\right\} = 2\gamma - \ln\left(\lim \frac{a_{n+1}}{n^2}\right)$$

$$\overset{s-c}{=} 2\gamma - \ln\left\{\lim \frac{a_{n+2} - a_{n+1}}{(n+1)^2 - n^2}\right\} = 2\gamma - \ln\left(r \cdot \lim \frac{n+1}{2n+1}\right) = 2\gamma - \ln\left(\frac{r}{2}\right)$$

Solution 5 by Pham Duc Nam-Vietnam

If: $a_n > 0; r_n > 0, a_{n+1} = a_n + nr_n (\forall n \in \mathbb{N}^*)$ **and** $\lim\limits_{n\to\infty} r_n = r > 0$

Then find: $\Omega = \lim\limits_{n\to\infty}(2H_n - \ln(a_n))$

$$\lim_{n\to\infty} r_n = r \Rightarrow \lim_{n\to\infty} r_{n+1} = r;\ \lim_{n\to\infty} \frac{\sum_{k=1}^{n} kr_k}{n^2} = \frac{r}{2}$$

Indeed, by Stolz – Cesaro theorem: $\lim\limits_{n\to\infty} \frac{\sum_{k=1}^{n+1} kr_k - \sum_{k=1}^{n} kr_k}{(n+1)^2 - n^2} = \lim\limits_{n\to\infty} \frac{(n+1)r_{n+1}}{2n+1} = \frac{r}{2}$

For $k\le n, a_{k+1}-a_k=kr_k\Rightarrow \underbrace{\sum_{k=1}^{n}(a_{k+1}-a_k)}_{Telescoping\ sum}=\sum_{k=1}^{n}kr_k\Leftrightarrow a_{n+1}-a_1=\sum_{k=1}^{n}kr_k$

$$\Leftrightarrow a_{n+1}-a_n+a_n-a_1=\sum_{k=1}^{n}kr_k\Leftrightarrow a_n=\sum_{k=1}^{n}kr_k+a_1-nr_n\Rightarrow$$

$$\Rightarrow \frac{a_n}{n^2}=\frac{\sum_{k=1}^{n}kr_k}{n^2}+\frac{a_1}{n^2}-\frac{r_n}{n}\Rightarrow \ln(a_n)=\ln\left(\frac{\sum_{k=1}^{n}kr_k}{n^2}+\frac{a_1}{n^2}-\frac{r_n}{n}\right)+2\ln(n)$$

$$\Omega=\lim_{n\to\infty}(2H_n-\ln(a_n))=\lim_{n\to\infty}\left(2H_n-2\ln(n)-\ln\left(\frac{\sum_{k=1}^{n}kr_k}{n^2}+\frac{a_1}{n^2}-\frac{r_n}{n}\right)\right)=$$

$$=\lim_{n\to\infty}\left(2H_n-2\ln(n)-\ln\left(\frac{r}{2}\right)\right)$$

By definition: $\gamma=\lim_{n\to\infty}(H_n-\ln(n))\Rightarrow\Omega=2\gamma-\ln\left(\frac{r}{2}\right)$

UP.529 If $a_n>0; n\in\mathbb{N}^*; \lim_{n\to\infty}(a_{n+1}-a_n)=a>0$ then find:

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{\sqrt[n]{n!}}-\frac{1}{\sqrt[n+1]{(n+1)!}}\right)\cdot a_n^2$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

Solution 1 by proposers

We will use Lalescu's sequence:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\right)=\frac{1}{e}$$

$$\lim_{n\to\infty}\frac{a_n}{n}\overset{CESARO-STOLZ}{=}\lim_{n\to\infty}\frac{a_{n+1}-a_n}{n+1-n}=\lim_{n\to\infty}(a_{n+1}-a_n)=a$$

$$\Omega=\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}}{\sqrt[n]{n!}\cdot\sqrt[n+1]{(n+1)!}}\right)\cdot a_n^2=$$

$$=\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\right)\cdot\frac{n}{\sqrt[n]{n!}}\cdot\frac{n+1}{\sqrt[n+1]{(n+1)!}}\cdot\left(\frac{a_n}{n}\right)^2\cdot\frac{n}{n+1}=$$

$$=\frac{1}{e}\cdot\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}\cdot\lim_{n\to\infty}\frac{n+1}{\sqrt[n+1]{(n+1)!}}\cdot\lim_{n\to\infty}\left(\frac{a_n}{n}\right)^2\cdot\lim_{n\to\infty}\frac{n}{n+1}=\frac{1}{e}\cdot e\cdot e\cdot a^2\cdot 1=a^2e$$

Solution 2 by Angel Plaza-Spain

$\lim_{n\to\infty}\frac{a_{n+1}}{n+1}=\lim_{n\to\infty}(a_{n+1}-a_n)=a$. Therefore $\lim_{n\to\infty}\frac{a_n^2}{n^2}=\lim_{n\to\infty}\left(\frac{a_n}{n}\right)^2=a^2$. Hence

$$\Omega=a^2\cdot\lim_{n\to\infty}\left(\frac{1}{\sqrt[n]{n!}}-\frac{1}{\sqrt[n+1]{(n+1)!}}\right)\cdot n^2$$

By using that [1]:

$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}-\frac{n^2}{\sqrt[n]{n!}}\right)=e$, then, by Stirling formula for n!

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt[n]{n!}}-\frac{1}{\sqrt[n+1]{(n+1)!}}\right)\cdot n^2=-e+\lim_{n\to\infty}\frac{2n+1}{\sqrt[n+1]{(n+1)!}}$$

$$=-e+\lim_{n\to\infty}\frac{2n+1}{(n+1)e^{-1}}=-e+2e=e.$$

Therefore: $\Omega=a^2e$

Reference:

[1] D.M. Bătinețu – Giurgiu, Daniel Sitaru,Neculai Stanciu- 120 Years of Lalescu Sequences, RMM 29 (2021), Available at https://www.ssmrmh.ro/wp-content/uploads/2020/02/120-YEARS-OF-LALESCU-SEQUENCES.pdf

UP.530 Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{\sqrt[n]{(2n-1)!!}}-\frac{1}{\sqrt[n+1]{(2n+1)!!}}\right)\cdot e^{2H_n}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

Solution 1 by proposers

$$\lim_{n\to\infty}\frac{n}{\sqrt[n]{(2n-1)!!}}=\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{(2n-1)!!}}=\lim_{n\to\infty}\frac{(n+1)^{n+1}}{(2n+1)!!}\cdot\frac{(2n-1)!!}{n^n}=$$

$$=\lim_{n\to\infty}\frac{(n+1)^n}{n^n}\cdot\frac{(2n-1)!!}{(2n-1)!!}\cdot\frac{n+1}{2n+1}=\frac{e}{2}$$

$$\lim_{n\to\infty}\frac{e^{H_n}}{n}=\lim_{n\to\infty}e^{-\log n+H_n}=e^{\gamma}$$

$$\lim_{n\to\infty}\frac{e^{2H_n}}{n^2}=e^{2\gamma}$$

$$\Omega=\lim_{n\to\infty}\left(\sqrt[n+1]{(2n+1)!!}-\sqrt[n]{(2n-1)!!}\right)\cdot\frac{n}{\sqrt[n]{(2n-1)!!}}\cdot\frac{n+1}{\sqrt[n+1]{(2n+1)!!}}\cdot\frac{e^{2H_n}}{n^2}\cdot\frac{n+1}{n}=$$

$$=\frac{2}{e}\cdot\frac{e}{2}\cdot\frac{e}{2}\cdot e^{2\gamma}\cdot 1=\frac{1}{2}e^{2\gamma+1}$$

Solution 2 by Angel Plaza-Spain

$\lim\limits_{n\to\infty}\frac{e^{H_n}}{n}=\lim\limits_{n\to\infty}\frac{e^{H_n}}{e^{\ln n}}=\lim\limits_{n\to\infty}e^{H_n-\ln n}=e^{\gamma}$. Therefore, $\lim\limits_{n\to\infty}\frac{e^{2H_n}}{n^2}=\lim\limits_{n\to\infty}\left(\frac{e^{H_n}}{n}\right)^2=e^{2\gamma}$

Hence, $\Omega=e^{2\gamma}\cdot\lim\limits_{n\to\infty}\left(\frac{1}{\sqrt[n]{(2n-1)!!}}-\frac{1}{\sqrt[n+1]{(2n+1)!!}}\right)\cdot n^2$

By using that [1]

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{(2n+1)!!}}-\frac{n^2}{\sqrt[n]{(2n+1)!!}}\right)=\frac{e}{2}$$

Then,

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt[n]{(2n-1)!!}}-\frac{1}{\sqrt[n+1]{(2n+1)!!}}\right)\cdot n^2=-\frac{e}{2}+\lim_{n\to\infty}\frac{2n+1}{\sqrt[n+1]{(2n+1)!!}}$$

$$=-\frac{e}{2}+\lim_{n\to\infty}\frac{(2n+1)\sqrt[n+1]{(2n+2)!!}}{\sqrt[n+1]{(2n+2)!}}=-\frac{e}{2}+\lim_{n\to\infty}\frac{(2n+1)\sqrt[n+1]{2^{n+1}(n+1)^{n+1}e^{-n-1}}}{\sqrt[n+1]{(2n+2)^{2n+2}e^{-2n-2}}}$$

$$=-\frac{e}{2}+\lim_{n\to\infty}\frac{(2n+1)2(n+1)e^{-1}}{(2n+2)^2e^{-2}}=-\frac{e}{2}+e=\frac{e}{2}$$

Therefore:

$$\Omega=e^{2\gamma}\cdot\frac{e}{2}=\frac{1}{2}e^{1+2\gamma}$$

Reference:

[1] D.M. Bătinețu - Giurgiu, Daniel Sitaru,Neculai Stanciu- 120 Years of Lalescu Sequences, RMM 29 (2021). Available at https://www.ssmrmh.ro/wp-content/uploads/2020/02/120-YEARS-OF-LALESCU-SEQUENCES.pdf

UP.531 Prove that:

$$\int_0^{\infty}\frac{x^2\sinh(2x)}{\cosh^2(2x)}dx=\frac{3\pi^3}{128}$$

Proposed by Said Attaoui – Algeria

Solution by proposer

The following problems are crucial to our solution

Problem 1. Prove

$$\beta(3)=\frac{\pi^3}{32},$$

with $\beta(s)$ design the Dirichlet beta function defined as

$$\beta(s)=\sum_{k=0}^{\infty}\frac{(-1)^n}{(2n+1)^s},\mathcal{R}(s)>0.$$

Problem 2. Prove

$$\int_0^{\infty}\frac{x^2}{\cosh(x)}dx=\frac{\pi^3}{8}.$$

Proof of Problem 1. We have

$$\beta(3)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^{2n}}{(2(2n)+1)^3}+\sum_{n=0}^{\infty}\frac{(-1)^{2n+1}}{(2(2n+1)+1)^3}=\frac{1}{64}\left(\sum_{n=0}^{\infty}\frac{1}{\left(n+\frac{1}{4}\right)^3}-\sum_{n=0}^{\infty}\frac{1}{\left(n+\frac{3}{4}\right)^3}\right)$$

$$=-\frac{1}{128}\left(\psi^{(2)}\left(\frac{1}{4}\right)-\psi^{(2)}\left(\frac{3}{4}\right)\right),$$

where $\psi(x)$ is the digamma function and $\psi^{(k)}(x)$ is its derivatives of order $k\geq 1$ given as:

$$\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}-\gamma+\sum_{n=0}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+x}\right)\ x>0;\ \psi^{(k)}(x)=(-1)^{k+1}k!\sum_{n=0}^{\infty}\frac{1}{(n+x)^{k+1}}.$$

Applying a reflection formula for the general $k-$ th order polygamma function $\psi^{(k)}(x)$, that is

$$\psi^{(k)}(1-x)+(-1)^{k+1}\psi^{(k)}(x)=(-1)^k\pi\frac{\partial^k}{\partial x^k}\cot(\pi x),$$

we can deduce for $x=\frac{3}{4}$:

$$\psi^{(2)}\left(\frac{1}{4}\right)-\psi^{(2)}\left(\frac{3}{4}\right)=\pi\left[\frac{\partial^2}{\partial x^2}\left(\frac{\cos(\pi x)}{\sin(\pi x)}\right)\right]_{x=\frac{3}{4}}=\pi\left[\frac{\partial}{\partial x}\left(\frac{-\pi}{\sin^2(\pi x)}\right)\right]_{x=\frac{3}{4}}$$

$$=\pi\left[\frac{2\pi^2\cot(\pi x)}{\sin^2(\pi x)}\right]_{x=\frac{3}{4}}=\frac{2\pi^3(-1)}{\left(\frac{1}{\sqrt{2}}\right)^2}=-4\pi^3$$

Thereby:

$$\beta(3)=-\frac{1}{128}(-4\pi^3)=\frac{\pi^3}{32}.$$

Remark. It's worth noting that the result of Problem 1 exist without proof in mathworld.wolfram.com (1) and wikipedia.org (2)

(1) https://mathworld.wolfram.com/DirichletBetaFunction.html

(2) https://en.wikipedia.org/wiki/Dirichlet_beta_function

Proof of Problem 2.

Let $J=\int_0^\infty \frac{x^2}{\cosh(x)}dx$, **so we have**

$$J=2\int_0^\infty \frac{x^2}{e^x+e^{-x}}dx=2\int_0^\infty \frac{x^2e^{-x}}{1+e^{-2x}}dx=2\int_0^\infty x^2e^{-x}\left(\sum_{n=0}^\infty(-1)^n e^{-2nx}\right)dx$$

$$=2\left(\sum_{n=0}^\infty(-1)^n\int_0^\infty x^2e^{-(2n+1)x}\,dx\right)=2\left(\sum_{n=0}^\infty\frac{(-1)^n}{(2k+1)^3}\int_0^\infty t^2e^{-t}\,dt\right)$$

$$=2\left(\sum_{n=0}^\infty\frac{(-1)^n\Gamma(3)}{(2n+1)^3}\right)=4\left(\sum_{n=0}^\infty\frac{(-1)^n}{(2n+1)^3}\right)=4\beta(3)=4\left(\frac{\pi^3}{32}\right)=\frac{\pi^3}{8}$$

Now, replacing x **by** $ax, a>0$ **in the integral** J**, we obtain**

$$\frac{\pi^3}{8}=\int_0^\infty\frac{(ax)^2}{\cosh(ax)}a\,dx,$$

which imply

$$\frac{\pi^3}{8a^3}=\int_0^\infty\frac{x^2}{\cosh(ax)}dx.$$

Differentiate both sides with respect to a**, we get**

$$-\frac{3\pi^3}{8a^3}=\int_0^\infty x^2\frac{\partial}{\partial a}\left(\frac{1}{\cosh(ax)}\right)dx=\int_0^\infty x^2\left(-\frac{x\sinh(ax)}{\cosh^2(ax)}\right)dx$$

Hence

$$\frac{3\pi^3}{8a^4} = \int_0^\infty \frac{x^2 \sinh(ax)}{\cosh^2(ax)} dx$$

Fix $a = 2$, we finally have

$$\int_0^\infty \frac{x^2 \sinh(2x)}{\cosh^2(2x)} dx = \frac{3\pi^3}{128}.$$

UP.532 Find all continuous functions $f: \mathbb{R} \to \mathbb{R}$; $f(0) = 0$ such that:

$$f(x) = f\left(\frac{x}{5}\right) + \frac{x}{7}; (\forall) x \in \mathbb{R}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

Replacing x successively with: $\frac{x}{5}; \frac{x}{5^2}; \dots; \frac{x}{5^{n-1}}$

$$f(x) - f\left(\frac{x}{5}\right) = \frac{x}{7}$$

$$f\left(\frac{x}{5}\right) - f\left(\frac{x}{5^2}\right) = \frac{x}{7 \cdot 5}$$

$$f\left(\frac{x}{5^2}\right) - f\left(\frac{x}{5^3}\right) = \frac{x}{7 \cdot 5^2}$$

— — — — — — — — — —

$$f\left(\frac{x}{5^{n-1}}\right) - f\left(\frac{x}{5^n}\right) = \frac{x}{7 \cdot 5^{n-1}}$$

By adding:

$$f(x) - f\left(\frac{x}{5^n}\right) = \frac{x}{7}\left(1 + \frac{1}{5} + \frac{1}{5^2} + \dots + \frac{1}{5^{n-1}}\right)$$

$$f(x) - f\left(\frac{x}{5^n}\right) = \frac{x}{7} \cdot \frac{\frac{1}{5^n} - 1}{\frac{1}{5} - 1}$$

$$\lim_{n\to\infty}\left(f(x) - f\left(\frac{x}{5^n}\right)\right) = \lim_{n\to\infty} \frac{x}{7} \cdot \frac{\frac{1}{5^n} - 1}{\frac{1}{5} - 1}$$

$$f(x) - f(0) = \frac{x}{7} \cdot \frac{0 - 1}{-\frac{4}{5}}, \qquad f(x) - 0 = \frac{5x}{28} \Rightarrow f(x) = \frac{5x}{28}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $n \in \mathbb{N}$. For all $k \in \{0, 1, \dots, n-1\}$, if we replace x by $\frac{x}{5^k}$, we obtain

$$f\left(\frac{x}{5^k}\right) - f\left(\frac{x}{5^{k+1}}\right) = \frac{x}{7.5^k}, \text{then } \sum_{k=0}^{n-1}\left(f\left(\frac{x}{5^k}\right) - f\left(\frac{x}{5^{k+1}}\right)\right) = \sum_{k=0}^{n-1}\frac{x}{7.5^k}$$

$$\text{or } f(x) = f\left(\frac{x}{5^n}\right) + \frac{x}{7}.\frac{1-\left(\frac{1}{5}\right)^n}{\frac{4}{5}}, \forall x \in \mathbb{R}.$$

Since f is continuous on $\mathbb{R}$, then $\lim\limits_{n\to\infty} f\left(\frac{x}{5^n}\right) = f\left(\lim\limits_{n\to\infty}\frac{x}{5^n}\right) = f(0) = 0.$

If we tend n to infinity in the last relation and since $\lim\limits_{n\to\infty}\left(\frac{1}{5}\right)^n = 0$, we obtain

$$f(x) = \frac{5x}{28}, \quad \forall x \in \mathbb{R},$$

The function satisfies the conditions of the problem.

UP.533 Calculate the integral:

$$\int_{-1}^{1}\frac{\arccos x}{\sqrt{4x^4 + x^2 + 4}}dx$$

Proposed by Vasile Mircea Popa – Romania

Solution by proposer

We make the notation: $A = \int_{-1}^{1}\frac{\arccos x}{\sqrt{4x^4+x^2+4}}dx.$

We also consider the integral: $B = \int_{-1}^{1}\frac{\arcsin x}{\sqrt{4x^4+x^2+4}}dx$

We have:

$$A + B = \int_{-1}^{1}\frac{\arccos x + \arcsin x}{\sqrt{4x^4 + x^2 + 4}}dx = \frac{\pi}{2}\int_{-1}^{1}\frac{1}{\sqrt{4x^4 + x^2 + 4}}dx$$

But, we have:

$$\int_{-1}^{1}\frac{1}{\sqrt{4x^4 + x^2 + 4}}dx = 2\int_{0}^{1}\frac{1}{\sqrt{4x^4 + x^2 + 4}}dx$$

because the function under the integral sign is even.

We are going to calculate the integral:

$$C=\int_0^1 \frac{1}{\sqrt{4x^4+x^2+4}}dx$$

We will show that the C integral can be expressed using the complete elliptic integral of the first kind.

The complete elliptic integral of the first kind is defined by the relationship:

$$K(k)=\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-k^2\sin^2\theta}}d\theta, \text{ with } -1<k<1.$$

Substitute:

$$t=\tan\frac{\theta}{2}, \text{ so } \sin\theta=\frac{2t}{1+t^2} \text{ and } d\theta=\frac{2}{1+t^2}dt.$$

We have:

$$K(k)=\int_0^1 \frac{1}{\sqrt{1-k^2\frac{4t^2}{(1+t^2)^2}}}2\frac{1}{1+t^2}dt=2\int_0^1 \frac{1}{\sqrt{t^4+(2-4k^2)t^2+1}}dt$$

We have:

$$C=\frac{1}{2}\int_0^1 \frac{1}{\sqrt{x^4+\frac{1}{4}x^2+1}}dx$$

We put the condition: $2-4k^2=\frac{1}{4}$, so $k=\frac{\sqrt{7}}{4}$ (we choose $k>0$).

We obtain: $C=\frac{1}{2}\cdot\frac{1}{2}\cdot K\left(\frac{\sqrt{7}}{4}\right)$

The integral B is equal to zero, because the function under the integral sign is odd.

So, we have: $A=\frac{\pi}{2}\cdot 2C$

We obtained the value of the integral required in the problem statement:

$$A=\frac{\pi}{4}K\left(\frac{\sqrt{7}}{4}\right).$$

UP.534 If $a,b>0$ then:

$$\int_a^b\int_a^b\left(\frac{x}{x^4+y^2}+\frac{y}{y^4+x^2}\right)dxdy\le \ln^2\left(\frac{b}{a}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$\frac{x}{x^4+y^2}+\frac{y}{y^4+x^2}\overset{AM-GM}{\leq}\frac{x}{2\sqrt{x^4y^2}}+\frac{y}{2\sqrt{y^4x^2}}=$$

$$=\frac{x}{2x^2y}+\frac{y}{2y^2x}=\frac{1}{2xy}+\frac{1}{2xy}=\frac{1}{xy}$$

$$\int_a^b\int_a^b\left(\frac{x}{x^4+y^2}+\frac{y}{y^4+x^2}\right)dxdy\leq\int_a^b\int_a^b\frac{1}{xy}dxdy=$$

$$=\left(\int_a^b\frac{1}{x}dx\right)\left(\int_a^b\frac{1}{y}dy\right)=(\ln b-\ln a)^2=\ln^2\left(\frac{b}{a}\right)$$

Equality holds for $a=b$.

Solution 2 by Tapas Das – India

$$\frac{x}{x^4+y^2}\overset{AM-GM}{\leq}\frac{x}{2\sqrt{x^4y^2}}=\frac{1}{2xy}$$

$$\frac{y}{y^4+x^2}\leq\frac{y}{2\sqrt{y^4x^2}}=\frac{1}{2xy}$$

$$\therefore\frac{x}{x^4+y^2}+\frac{y}{y^4+x^2}\leq\frac{1}{2xy}+\frac{1}{2xy}=\frac{1}{xy}$$

$$\int_a^b\int_a^b\left[\left(\frac{x}{x^4+y^2}\right)+\left(\frac{y}{y^4+x^2}\right)\right]dx\,dy\leq\int_a^b\int_a^b\frac{1}{xy}dx\,dy$$

$$=\int_a^b\frac{1}{x}dx\cdot\int_a^b\frac{1}{y}dy=(\ln b-\ln a)^2=\ln^2\frac{b}{a}$$

UP.535 If $x,y,z>1;x\neq y\neq z\neq x$ and

$$\log_{\frac{y}{z}}x+\log_{\frac{z}{x}}y+\log_{\frac{x}{y}}z=0$$

then:

$$\frac{\log_2 x}{\log_2^2\left(\frac{y}{z}\right)}+\frac{\log_2 y}{\log_2^2\left(\frac{z}{x}\right)}+\frac{\log_2 z}{\log_2^2\left(\frac{x}{y}\right)}=0$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$0=\sum_{cyc}\log_{\frac{y}{z}}x=\left(\sum_{cyc}\log_{\frac{y}{z}}x\right)\cdot\left(\sum_{cyc}\frac{1}{\log_2\frac{y}{z}}\right)=$$

$$=\left(\sum_{cyc}\frac{\log_2 x}{\log_2 y-\log_2 z}\right)\cdot\left(\sum_{cyc}\frac{1}{\log_2 y-\log_2 z}\right)=$$

$$=\sum_{cyc}\frac{\log_2 x}{(\log_2 y-\log_2 z)^2}+\sum_{cyc}\frac{\log_2 y+\log_2 z}{(\log_2 z-\log_2 x)(\log_2 x-\log_2 y)}=$$

$$=\sum_{cyc}\frac{\log_2 x}{\log_2^2\left(\frac{y}{z}\right)}+\sum_{cyc}\frac{(\log_2 y+\log_2 z)(\log_2 y-\log_2 z)}{(\log_2 z-\log_2 x)(\log_2 x-\log_2 y)(\log_2 y-\log_2 z)}=$$

$$=\sum_{cyc}\frac{\log_2 x}{\log_2^2\left(\frac{y}{z}\right)}+\frac{1}{\prod_{cyc}(\log_2 z-\log_2 x)}\cdot\sum_{cyc}(\log_2^2 y-\log_2^2 z)=$$

$$=\sum_{cyc}\frac{\log_2 x}{\log_2^2\left(\frac{y}{z}\right)}+\frac{1}{\prod_{cyc}(\log_2 z-\log_2 x)}\cdot 0=\sum_{cyc}\frac{\log_2 x}{\log_2^2\left(\frac{y}{z}\right)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $a=\ln x, b=\ln y, c=\ln z$.

The given condition is equivalent to $\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}=0$,

and the problem becomes to prove that $\left(\frac{a}{(b-c)^2}+\frac{b}{(c-a)^2}+\frac{c}{(a-b)^2}\right).\ln 2=0$.

We have

$$\frac{a}{(b-c)^2}+\frac{b}{(c-a)^2}+\frac{c}{(a-b)^2}=$$

$$=\left(\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}\right)\left(\frac{1}{b-c}+\frac{1}{c-a}+\frac{1}{a-b}\right)-$$

$$-\frac{a}{b-c}\left(\frac{1}{c-a}+\frac{1}{a-b}\right)-\frac{b}{c-a}\left(\frac{1}{a-b}+\frac{1}{b-c}\right)-\frac{c}{a-b}\left(\frac{1}{b-c}+\frac{1}{c-a}\right)=0$$

$$\left(\frac{1}{b-c}+\frac{1}{c-a}+\frac{1}{a-b}\right)-\frac{a(c-b)+b(a-c)+c(b-a)}{(a-b)(b-c)(c-a)}=0,$$

which completes the proof.

UP.536 If $1<a\le b; m\ge 1$ then:

$$\frac{(b-1)^{m+1}-(a-1)^{m+1}}{m+1}+b-a\le\frac{b^{m+1}-a^{m+1}}{m+1}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$\frac{(b-1)^{m+1}-(a-1)^{m+1}}{m+1}=\int_a^b(x-1)^m\,dx$$

$$b-a=\int_a^b dx$$

$$\frac{b^{m+1}-a^{m+1}}{m+1}=\int_a^b x^m\,dx$$

We must prove that

$$\int_a^b(x-1)^m\,dx+\int_a^b dx\le\int_a^b x^m\,dx$$

$$\int_a^b((x-1)^m+1)\,dx\le\int_a^b x^m\,dx$$

It is enough to prove that:

$$(x-1)^m+1\le x^m;\ (\forall)x>1;\ m\ge 1$$

Let be: $f:(1,\infty)\to\mathbb{R};\ f(x)=(x-1)^m-x^m+1$

$f'(x)=m((x-1)^{m-1}-x^{m-1})<0$ because $x-1<x$

f decreasing on $(1,\infty)$

$$\sup_{x>1}f(x)=\lim_{\substack{x\to1\\x>1}}f(x)=\lim_{\substack{x\to1\\x>1}}((x-1)^m-x^m+1)=0$$

$$\Rightarrow f(x)\le 0;\ (\forall)x>1\Rightarrow (x-1)^m-x^m+1\le 0$$

Equality holds for $a=b$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Write the given inequality as $f(a)\le f(b)$, where

$$f(x)=\frac{x^{m+1}-(x-1)^{m+1}}{m+1}-x,\quad x\ge 1.$$

We have $f'(x)=x^m-(x-1)^m-1$ and

$f''(x)=m[x^{m-1}-(x-1)^{m-1}]\ge 0$, then f' is increasing on $[1,\infty)$ and $f'(x)\ge f'(1)=0, \forall x\ge 1$, then f is increasing on $[1,\infty)$, and since $1<a\le b$, then $f(a)\le f(b)$, as desired.

Equality holds if $a=b$.

UP.537 Find:

$$\Omega=\lim_{n\to\infty}\frac{1}{n\cdot 8^n}\binom{4n}{n}\binom{4n}{2n}\binom{3n}{n}^{-2}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Adrian Popa-Romania

$$\Omega=\lim_{n\to\infty}\frac{1}{n\cdot 8^n}\binom{4n}{n}\binom{4n}{2n}\binom{3n}{n}^{-2}=\lim_{n\to\infty}\frac{1}{n\cdot 8^n}\cdot\frac{(4n)!\cdot(4n)!}{n!\,(3n)!\,(2n)!\,(2n)!}\cdot\frac{(n!)^2\left((2n)!\right)^2}{\left((3n)!\right)^2}=$$

$$=\lim_{n\to\infty}\frac{1}{n\cdot 8^n}\cdot\frac{\left((4n)!\right)^2\cdot n!}{\left((3n)!\right)^3}=(*);\ but\ n!\cong\frac{n^n\sqrt{2n\pi}}{e^n}, then$$

$$(*)=\lim_{n\to\infty}\frac{[(4n)^{4n}]^2\cdot 2\cdot 4n\pi}{(e^{4n})^2}\cdot\frac{n^n\sqrt{2n\pi}}{e^n}\cdot\frac{(e^{3n})^3}{[(3n)^{3n}]^3\cdot 6n\pi\sqrt{6n\pi}}\cdot\frac{1}{n\cdot 8^n}=$$

$$=\lim_{n\to\infty}\frac{4^{2n}}{3^{9n}\cdot 3\sqrt{3n}\cdot 2^{3n}}=\lim_{n\to\infty}\frac{\left(\frac{4}{3}\right)^{8n}}{24^n\cdot 3\sqrt{3n}}=\lim_{n\to\infty}\left(\frac{\frac{4^8}{3^8}}{24}\right)^n\cdot\frac{1}{3\sqrt{3n}}=0.$$

Solution 2 by Ravi Prakash-New Delhi-India

$$Let\ a_n=\binom{4n}{n}\binom{4n}{2n}\binom{3n}{n}^{-2}=\frac{(4n)!\cdot(4n)!}{n!\,(3n)!\,(2n)!\,(2n)!}\cdot\frac{(n!)^2\left((2n)!\right)^2}{\left((3n)!\right)^2}=$$

$$=\frac{\left((4n)!\right)^2\cdot n!}{\left((3n)!\right)^3}=\frac{\left((3n+1)(3n+2)\dots(4n)\right)^2}{(2n+1)(2n+2)\dots(2n)}\le\frac{(4n)^{2n}}{(2n)^{2n}}=\frac{2^{4n}}{2^{2n}}=2^{2n}$$

$$Now, 0<\frac{1}{8^n\cdot n}a_n<\frac{1}{2^n\cdot n}. As\ \lim_{n\to\infty}\frac{1}{2^n\cdot n}=0, then:$$

$$\Omega=\lim_{n\to\infty}\frac{1}{n\cdot 8^n}\binom{4n}{n}\binom{4n}{2n}\binom{3n}{n}^{-2}=0.$$

UP.538 Solve for real numbers:

$$\int_4^x \frac{t^2+1}{t^3+1}dt = 2(\sqrt{x}-2)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Aki Le-Vietnam

$$Let\ f(x) = \int_4^x \frac{t^2+1}{t^3+1}dt - 2(\sqrt{x}-2),\ then$$

$$f'(x) = \frac{-(\sqrt{x}-1)(\sqrt{x^5}-1)}{(x^3+1)\sqrt{x}} \le 0, \forall x > 0$$

$$Moreover, f(4) = 0. Then\ the\ solution\ set\ is\ \{4\}.$$

Solution 2 by Christos Tsifakis-Greece

$$Let\ f(x) = \int_4^x \frac{t^2+1}{t^3+1}dt - 2(\sqrt{x}-2), x \ge 0.$$

$$For\ x = 4, we\ have\ f(4) = 0.$$

$$For\ x > 0, we\ have{:}\ f'(x) = \frac{x^2+1}{x^3+1} - \frac{1}{\sqrt{x}} = \frac{(x^2\sqrt{x}-1)(1-\sqrt{x})}{\sqrt{x}(x^3+1)} =$$

$$= \frac{-(\sqrt{x}-1)^2(\sqrt{x^4}+\sqrt{x^3}+\sqrt{x^2}+1)}{\sqrt{x}(x^3+1)} \le 0.$$

$$So, f \searrow [0,\infty)\ and\ x = 4\ is\ the\ only\ solution.$$

Solution 3 by Marin Chirciu – Romania

The condition of existence of the radical is $x \ge 0$

We decompose in simple fractions $\frac{t^2+1}{t^3+1} = \frac{A}{t+1} + \frac{Bt+C}{t^2-t+1}$

We obtain $A = \frac{2}{3}, B = \frac{1}{3}, C = \frac{1}{3}$

$$\int \frac{t^2+1}{t^3+1}dt = \frac{2}{3}\ln(t+1) + \frac{1}{3}\int \frac{t+1}{t^2-t+1}dt$$

$$\int \frac{t+1}{t^2-t+1}dt = \frac{1}{2}\int \frac{2t-1+3}{t^2-t+1}dt = \frac{1}{2}\int \frac{2t-1}{t^2-t+1}dt + \frac{3}{2}\int \frac{1}{t^2-t+1}dt =$$

$$= \frac{1}{2}\ln(t^2-t+1) + \frac{3}{2}\int \frac{1}{t^2-t+\frac{1}{4}+\frac{3}{4}}dt = \frac{1}{2}\ln(t^2-t+1) + \frac{3}{2}\int \frac{1}{\left(t-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dt =$$

$$=\frac{1}{2}\ln(t^2-t+1)+\frac{3}{2}\cdot\frac{1}{\frac{\sqrt{3}}{2}}\arctan\frac{t-\frac{1}{2}}{\frac{\sqrt{3}}{2}}=\frac{1}{2}(t^2-t+1)+\sqrt{3}\arctan\frac{2t-1}{\sqrt{3}}.$$

It follows:

$$\int\frac{t^2+1}{t^3+1}dt=\frac{2}{3}\ln(t+1)+\frac{1}{3}\int\frac{t+1}{t^2-t+1}dt=$$

$$=\frac{2}{3}\ln(t+1)+\frac{1}{3}\left(\frac{1}{2}\ln(t^2-t+1)+\sqrt{3}\arctan\frac{2t-1}{\sqrt{3}}\right)=$$

$$=\frac{2}{3}\ln(t+1)+\frac{1}{6}\ln(t^2-t+1)+\frac{\sqrt{3}}{3}\arctan\frac{2t-1}{\sqrt{3}}$$

We obtain:

$$\int_4^x\frac{t^2+1}{t^3+1}dt=\left[\frac{2}{3}\ln(t+1)+\frac{1}{6}\ln(t^2-t+1)+\frac{\sqrt{3}}{3}\arctan\frac{2t-1}{\sqrt{3}}\right]_4^x=$$

$$=\frac{2}{3}\ln\frac{x+1}{5}+\frac{1}{6}\ln\frac{x^2-x+1}{13}+\frac{\sqrt{3}}{3}\arctan\frac{\frac{2x-1}{\sqrt{3}}-\frac{7}{\sqrt{3}}}{1+\frac{2x-1}{\sqrt{3}}\cdot\frac{7}{\sqrt{3}}}=$$

$$=\frac{2}{3}\ln\frac{x+1}{5}+\frac{1}{6}\ln\frac{x^2-x+1}{13}+\frac{\sqrt{3}}{3}\arctan\frac{\frac{2x-8}{\sqrt{3}}}{1+\frac{7(2x-1)}{3}}.$$

The equation $\int_4^x\frac{t^2+1}{t^3+1}dt=2(\sqrt{x}-2)$ can be written:

$\frac{2}{3}\ln\frac{x+1}{5}+\frac{1}{6}\ln\frac{x^2-x+1}{13}+\frac{\sqrt{3}}{3}\arctan\frac{\frac{2x-8}{\sqrt{3}}}{1+\frac{7(2x-1)}{3}}=2(\sqrt{x}-2)$, with $x=4$ unique solution.

We deduce that $x=4$ is the unique solution of the equation.

Solution 4 by Angel Plaza-Spain

By simple inspection it follows that $x=4$ is a real solution to the given equation. Let us prove that there is no solution. Consider function $f(x)$ defined by

$$f(x)=\int_4^x\frac{t^2+1}{t^3+1}dt-2(\sqrt{x}-2),$$

for $x \geq 0$. f is a continuous function for $x \geq 0$. Then by the Fundamental Theorem of Integral Calculus, $f'(x) = \frac{x^2+1}{x^3+1} - \frac{1}{\sqrt{x}} = -\frac{(\sqrt{x}-1)^2(x^2+x\sqrt{x}+x+\sqrt{x}+1)}{\sqrt{x}(x+1)(x^2-x+1)} \leq 0$ for all $x \geq 0$.

Therefore, function $f(x)$ is monotonically decreasing and it has no more than a single root in $\mathbb{R}^+$.

UP.539 If $0 < a \leq b$ then:

$$\int_a^b e^{x^2}\,dx \geq (b-a)\cdot\sqrt[3]{a^2+ab+b^2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by George Florin Șerban – Romania

$$e^x \geq x+1, (\forall)x>0 \Rightarrow e^{x^2} \geq x^2+1$$

$$\Rightarrow \int_a^b e^{x^2}\,dx \geq \int_a^b (x^2+1)\,dx = \frac{b^3-a^3}{3} + b - a =$$

$$= \frac{(b-a)(b^2+ab+a^2)}{3} + b - a = (b-a)\cdot\left(\frac{a^2+ab+b^2}{3}+1\right) \geq$$

$$\geq (b-a)\cdot\sqrt[3]{a^2+ab+b^2}$$

$$b-a \geq 0, \frac{a^2+ab+b^2}{3}+1 \geq \sqrt[3]{a^2+ab+b^2}$$

$$S = a^2+ab+b^2 > 0$$

$$\frac{S+3}{3} \geq \sqrt[3]{S} \Rightarrow (S+3)^3 \geq 27S \Rightarrow S^3+9S^2+27S-27S \geq 0$$

$$\Rightarrow S^3+9S^2 > 0, \text{ true, } (\forall)S>0$$

Then

$$\int_a^b e^{x^2}\,dx \geq (b-a)\sqrt[3]{a^2+ab+b^2}$$

Solution 2 by Marin Chirciu – Romania

Using $e^t \geq t+1, t \geq 0$, for $t = x^2 \geq 0$ we obtain:

$$\int_a^b e^{x^2}\,dx \geq \int_a^b (x^2+1)\,dx = \left(\frac{x^3}{3}+x\right)\Bigg|_a^b = \frac{b^3-a^3}{3} + b - a =$$

$$= (b-a)\left(\frac{b^2+ba+a^2}{3}+1\right)\overset{(1)}{\geq}(b-a)\sqrt[3]{a^2+ab+b^2}$$

where (1) $\Leftrightarrow \frac{b^2+ba+a^2}{3}+1 \geq \sqrt[3]{a^2+ab+b^2}$, which follows from:

We denote $\sqrt[3]{a^2+ab+b^2}=t>0$ and the inequality $\frac{b^2+ba+a^2}{3}+1\geq\sqrt[3]{a^3+ab+b^2}$

can be written:

$\frac{t^3}{3}+1\geq 1 \Leftrightarrow t^3-3t+3\geq 0$, true from $t^3-3t+3\overset{(2)}{\geq}1>0$,

where (2) $\Leftrightarrow t^3-3t+3\geq 1 \Leftrightarrow t^3-3t+2\geq 0 \Leftrightarrow (t-1)^2(t+2)\geq 0$ with equality for $t=1$.

Equality holds if and only if $a=b$.

Solution 3 by Hikmat Mammadov – Azerbaijan

The function $f: x\to e^{x^2}$ is convex so $\forall x\in[a,b]$

$$f(x)\geq f\left(\frac{a+b}{2}\right)+f'\left(\frac{a+b}{2}\right)\left(x-\frac{a+b}{2}\right)$$

So (by integrations on $[a,b]$)

$$\int_a^b f(x)\,dx\geq(b-a)f\left(\frac{a+b}{2}\right)$$

The derivative function of $g: t\to e^{\frac{3t}{4}}-t$ is $g': t\to\frac{3}{4}e^{\frac{3t}{4}}-1$

So the minimum of the function g is $g\left(\frac{4}{3}\ln\left(\frac{4}{3}\right)\right)=\frac{4}{3}\left(1-\ln\left(\frac{4}{3}\right)\right)\geq 0$

So $\forall t\in\mathbb{R}$ and $g(t)\geq 0$

With $t=(a+b)^2$ and we get $e^{\frac{3}{4}(a+b)^2}\geq(a+b)^2$

Since $(a+b)^2\geq a^2+b^2+ab$ and we get $e^{3\left(\frac{a+b}{2}\right)^2}\geq a^2+b^2+ab$

So $e^{\left(\frac{a+b}{2}\right)^2}\geq\sqrt[3]{a^2+b^2+ab}$

That gives $(b-a)f\left(\frac{a+b}{2}\right)\geq(b-a)\sqrt[3]{a^2+b^2+ab}$

And finally:

$$\int_a^b e^{x^2}\,dx \ge (b-a)\sqrt[3]{a^2+b^2+ab}$$

UP.540 If $a,b\in\mathbb{R}, a<b, f:[a,b]\to(0,\infty), f-$ continuous then:

$$3\int_a^b f(x)\,dx+\frac{1}{(b-a)^2}\left(\int_a^b \frac{1}{f(x)}\,dx\right)^3 \ge 4(b-a)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

For $x,y,z\in[a,b], f(x),f(y),f(z)>0$

$$f(x)+f(y)+f(z)+\frac{1}{f(x)\cdot f(y)\cdot f(z)} \overset{AM-GM}{\ge} 4\sqrt[4]{f(x)f(y)f(z)\cdot\frac{1}{f(x)f(y)f(z)}}=4$$

$$\int_a^b\int_a^b\int_a^b\left(f(x)+f(y)+f(z)+\frac{1}{f(x)\cdot f(y)\cdot f(z)}\right)dxdydz \ge \int_a^b\int_a^b\int_a^b 4dxdydz$$

$$3\int_a^b f(x)dx\cdot(b-a)^2+\left(\int_a^b\frac{1}{f(x)}dx\right)^3 \ge 4(b-a)^3$$

$$3\int_a^b f(x)\,dx+\frac{1}{(b-a)^2}\left(\int_a^b \frac{1}{f(x)}\,dx\right)^3 \ge 4(b-a)$$

Solution 2 by Marin Chirciu – Romania

With CBS inequality we have:

$$\int_a^b f(x)\,dx\cdot\int_a^b\frac{1}{f(x)}dx \ge \left(\int_a^b 1\,dx\right)^2=(b-a)^2 \Rightarrow$$

$$\Rightarrow \int_a^b f(x)\,dx\cdot\int_a^b\frac{1}{f(x)}dx \ge (b-a)^2 \Rightarrow$$

$$\Rightarrow \int_a^b f(x)\,dx \ge \frac{(b-a)^2}{\int_a^b\frac{1}{f(x)}dx} \quad (1)$$

Using (1) it suffices to prove that:

$$3\cdot\frac{(b-a)^2}{\int_a^b\frac{1}{f(x)}dx}+\frac{1}{(b-a)^2}\left(\int_a^b\frac{1}{f(x)}dx\right)^3 \ge 4(b-a)$$

which follows from:

We denote $\int_a^b \frac{1}{f(x)} dx = I$ and the above inequality can be written

$$3 \cdot \frac{(b-a)^2}{I} + \frac{1}{(b-a)^2} I^3 \geq 4(b-a) \Leftrightarrow$$

$$\Leftrightarrow I^4 - 4(b-a)^3 I + 3(b-a)^4 \geq 0 \Leftrightarrow$$

$$\Leftrightarrow [I-(b-a)^2]^2[I^2 + 2(b-a)I + 3(b-a)^2] \geq 0$$

with equality for $I = b - a$.