PROBLEMS FOR JUNIORS

JP.526. Let $a, b, c \in (0,1)$. Prove that:

$$\frac{b+c}{1-a}+\frac{c+a}{1-b}+\frac{a+b}{1-c} \geq \frac{2\sqrt{ab}}{1-\sqrt{ab}}+\frac{2\sqrt{bc}}{1-\sqrt{bc}}+\frac{2\sqrt{ca}}{1-\sqrt{ca}}$$

Proposed by Marin Chirciu - Romania

JP.527. If $x, y, z > 0$ with $x+y+z=1$ and $\lambda \geq 21$ then:

$$\frac{1}{x^3+y^3+z^3}+\frac{\lambda}{xy+yz+zx} \geq 3(\lambda+3)$$

Proposed by Marin Chirciu - Romania

JP.528. If $a, b, c > 0$ such that $a+b+c=3$ and $0 \leq \lambda \leq \frac{1}{2}$ then:

$$\frac{1}{a^3+\lambda}+\frac{1}{b^3+\lambda}+\frac{1}{c^3+\lambda} \geq \frac{3}{\lambda+1}$$

Proposed by Marin Chirciu - Romania

JP.529. If $a, b, c > 0; x \in \mathbb{R}$ then:

$$\frac{a}{(b\sin^2 x+c\cos^2 x)^3}+\frac{b}{(c\sin^2 x+b\cos^2 x)^3}+$$
$$+\frac{c}{(a\sin^2 x+c\cos^2 x)^3} \geq \frac{27}{(a+b+c)^2}$$

Proposed by Daniel Sitaru - Romania

JP.530. In $\Delta ABC, O$ - circumcenter. A_1, B_1, C_1 are the intersection points of AO, BO, CO with BC, AC and AB respectively. R_1, R_2 and R_3 are circumradii of $\Delta BOC, \Delta AOC$ and ΔAOB respectively. Show that:

$$R\left(\frac{1}{OA_1}+\frac{1}{OB_1}+\frac{1}{OC_1}\right)+3=\frac{4F}{R^2}\left(\frac{R_1}{BC}+\frac{R_2}{AC}+\frac{R_3}{AB}\right)$$

Proposed by Ertan Yildirim - Turkiye

JP.531. If $a, b, c > 0$ and $abc=1$ then:

$$\left(\frac{a}{2}+\frac{b}{c}\right)^3+\left(\frac{b}{2}+\frac{c}{a}\right)^3+\left(\frac{c}{2}+\frac{a}{b}\right)^3 \geq 3\sqrt[4]{18}$$

Proposed by Khaled Abd Imouti - Syria

JP.532. In any $\Delta ABC, I$ - incenter, r - radii, R - circumradius, s - semiperimeter, the following relationship holds:

$$AB + BI + CI \leq 2(R + r)$$

Proposed by Marian Ursărescu - Romania

JP.533. Let be the triangle ABC with AD, BE, CF - altitudes and H - orthocenter. Prove that:

$$\frac{HA}{HD} + \frac{HB}{HE} + \frac{HC}{HF} \geq 2\left(\left(\frac{R}{r}\right)^2 - 1\right)$$

Proposed by Marian Ursărescu - Romania

JP.534. In $\Delta ABC, I$ - incenter and D, E, F the points of contact of the cevians AI, BI, CI with the circle, then the following relationship holds:

$$ID + IE + IF \leq \frac{2(R^2 - Rr + r^2)}{r}$$

Proposed by Marian Ursărescu - Romania

JP.535. In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{r_a^4 + r_b^2 r_c^2}{r_b^2 + r_c^2} \geq s^2$$

Proposed by Marian Ursărescu - Romania

JP.536. In ΔABC the following relationship holds:

$$\frac{2R}{r} \geq \frac{(4R + r)^2}{s^2} + 1$$

Proposed by Alex Szoros - Romania

JP.537. Find the angles of a triangle ABC if

$$\frac{\sin A + 2\sin B}{\sqrt{\sin^2 B + \sin^2 C + 2\cos A \sin B \sin C}} + 1 = \frac{3\sqrt{3}}{2\sin C}$$

Proposed by Cristian Miu - Romania

JP.538. In ΔABC the following relationship holds:

$$\frac{3}{2R} \leq \sum \frac{\cos^2 \frac{A}{2}}{h_a} \leq \frac{3}{4r}$$

Proposed by Alex Szoros - Romania

JP.539. **In $\Delta ABC, O \in (AB), OQ \parallel BC$, where $Q \in (AC)$. $P \in (OC)$ such that $RP \parallel BC$, where $R \in (AC)$ and $T \in (AB)$. If the lengths of the segment RT is the geometric mean of the lengths of the segments OQ and BC, then $OP < \frac{OC}{2}$.**

Proposed by Gheorghe Molea - Romania

JP.540. **Let be ΔACD with $m(\widehat{CAD}) > 90^\circ, B \in (CD)$, such that $m(\widehat{BAC}) = 90^\circ$ and $AC > AB$. The bisector $\widehat{ACD}$ intersects AD in E. If $(BE$ is the bisector $\widehat{ABD}$, prove that:**

$$\frac{1}{AD} = \frac{\sqrt{2}}{2}\left(\frac{1}{AB} - \frac{1}{AC}\right)$$

Proposed by Gheorghe Molea - Romania

PROBLEMS FOR SENIORS

SP.526. **If $a, b, c, \lambda > 0, a + b + c = \lambda$ then:**

$$\sum \sqrt{\frac{bc}{a} + \lambda} \geq 2(\sqrt{a} + \sqrt{b} + \sqrt{c})$$

Proposed by Marin Chirciu - Romania

SP.527. **If $x, y, z \geq 0$ with $x + y + z = 1$ and $0 \leq \lambda \leq \frac{9}{4}$ then:**

$$xy + yz + zx - \lambda xyz \leq \frac{9 - \lambda}{27}$$

Proposed by Marin Chirciu - Romania

SP.528. **If $a, b, c \geq 1$ then:**

$$\frac{1}{9}(a + b + c) + \frac{1}{3\sqrt{2}} \geq \frac{\sqrt[3]{ab - 1}}{b + c + \sqrt{2}} + \frac{\sqrt[3]{bc - 1}}{c + a + \sqrt{2}} + \frac{\sqrt[3]{ca - 1}}{a + b + \sqrt{2}}$$

Proposed by Marin Chirciu - Romania

SP.529. **Let ABC be a triangle with inradius r and circumradius R and let the interior points D, E, F be chosen on the sides BC, CA, AB respectively, so that AD, BE, CF are the bisectors of the triangle ABC. Let r_A, r_B, r_C be the inradii of the triangles AEF, BFD, CDE respectively. Prove that:**

$$r_A^2 + r_B^2 + r_C^2 \leq \frac{3R^2}{64r^2}$$

Proposed by George Apostolopoulos – Greece

SP.530. Let a, b, c be the lengths of the sides of a triangle with inradius circumradius R. Prove that:

$$\frac{3^{1-\frac{1}{2n}}}{\sqrt[n]{(m+1)R}} \leq \frac{1}{\sqrt[n]{m \cdot a + b}} + \frac{1}{\sqrt[n]{m \cdot b + c}} + \frac{1}{\sqrt[n]{m \cdot c + a}} \leq \frac{3^{1-\frac{1}{2n}}}{\sqrt[n]{(m+1) \cdot 2r}}$$

for all integers $m \geq 0$ and $n \geq 1$.

Proposed by George Apostolopoulos – Greece

SP.531. If $a, b, c \geq 1$, then:

$$\sqrt{\frac{ab + bc + ca}{3}} - \sqrt[3]{abc} \geq \sqrt{\frac{\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}}{3}} - \frac{1}{\sqrt[3]{abc}}$$

Proposed by Vasile Mircea Popa - Romania

SP.532. Prove that in any right triangle with the cathetus b and c we have the inequality: $r \leq \frac{2-\sqrt{2}}{4}(b + c)$, where r is the inradii of the triangle.

Proposed by Laura Molea and Gheorghe Molea - Romania

SP.533. Prove that $k = \frac{4}{5}$ is the largest positive value of the constant k such that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} - 5 \geq k(a + b + c + d + e - 5)$$

for any positive real numbers a, b, c, d, e satisfying $ab + bc + cd + de + ea = 5$

Proposed by Vasile Cârtoaje - Romania

SP.534. If the lengths a, b, c of the sides of a triangle are the roots of the equation $kx^3 - lx^2 + 9kx - l = 0 \quad (k \cdot l \neq 0)$, then find the area of the triangle.

Proposed by George Apostolopoulos – Greece

SP.535. Determine all the numbers $\overline{abcd}$ such that:

$$1 + a + b + c + a \cdot b + b \cdot c + c \cdot a = a \cdot b \cdot c \cdot d$$

Proposed by Neculai Stanciu, Titu Zvonaru – Romania

SP.536. If $x \geq 0$ then:

$$\frac{2}{\sqrt{\pi}}\left(\int_0^x e^{-t^2} dt\right)^2 + \int_0^{2x} e^{-t^2} dt \geq 2\int_0^x e^{-t^2} dt$$

Proposed by Daniel Sitaru - Romania

SP.537. Solve for real numbers:

$$3e^x + 3e^{3x} + 1 = 4e^{2x} + 3 \cdot \sqrt[3]{e^{4x}}$$

Proposed by Daniel Sitaru - Romania

SP.538. In acute ΔABC the following relationship holds:

$$36 \le 4\left(\sum_{cyc} \tan A \tan B\right) \le 9 + \prod_{cyc} \tan^2 A$$

Proposed by Daniel Sitaru - Romania

SP.539. If $a > 0; f : \mathbb{R} \to \mathbb{R}$ is a function such that:

$$f\left(ax - \frac{1}{a}\right) \le ax \le f(x) - 1; (\forall) x \in \mathbb{R} \text{ then:}$$

$$f(2) + f(4) + f(8) > \frac{12\sqrt{a}}{a}$$

Proposed by Daniel Sitaru - Romania

SP.540. If $x, y \in [3, 4]; z, t \in [1, 12]$ then:

$$(x + y + z + t)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{t}\right) \le \frac{100}{3}$$

Proposed by Daniel Sitaru - Romania

UNDERGRADUATE PROBLEMS

UP.526. Prove the identity:

$$\int_0^\infty \frac{|\sin(x)|}{1 + x^2} dx = 1 - 2\sum_{n=1}^{\infty} \frac{1}{(4n^2 - 1)e^{2n}}$$

Proposed by Vasile Mircea Popa - Romania

UP.527. Prove the closed form:

$$\int_0^\infty \frac{\ln x}{x^3 + x\sqrt{x} + 1} dx = -\frac{32\pi^2}{81} \sin \frac{\pi}{18}$$

Proposed by Vasile Mircea Popa - Romania

UP.528. **If $a_n > 0; r_n > 0; a_{n+1} = a_n + n \cdot r_n; n \in \mathbb{N}^*$ and**

$$\lim_{n\to\infty} r_n = r > 0$$

then find:

$$\Omega = \lim_{n\to\infty}(2H_n - \log a_n)$$

Proposed by D.M. Bătineţu - Giurgiu, Daniel Sitaru - Romania

UP.529. **If $a_n > 0; n \in \mathbb{N}^*; \lim_{n\to\infty}(a_{n+1} - a_n) = a > 0$ then find:**

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{\sqrt[n]{n!}} - \frac{1}{\sqrt[n+1]{(n+1)!}}\right)\cdot a_n^2$$

Proposed by D.M. Bătineţu - Giurgiu, Daniel Sitaru - Romania

UP.530. **Find:**

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{\sqrt[n]{(2n-1)!!}} - \frac{1}{\sqrt[n+1]{(2n+1)!!}}\right)\cdot e^{2H_n}$$

Proposed by D.M. Bătineţu - Giurgiu, Daniel Sitaru - Romania

UP.531. **Prove that:**

$$\int_0^\infty \frac{x^2\sinh(2x)}{\cosh^2(2x)}dx = \frac{3\pi^3}{128}$$

Proposed by Said Attaoui – Algeria

UP.532. **Find all continuous functions $f : \mathbb{R} \to \mathbb{R}; f(0) = 0$ such that:**

$$f(x) = f\left(\frac{x}{5}\right) + \frac{x}{7}; (\forall)x \in \mathbb{R}$$

Proposed by Daniel Sitaru - Romania

UP.533. **Calculate the integral:**

$$\int_{-1}^1 \frac{\arccos x}{\sqrt{4x^4 + x^2 + 4}}dx$$

Proposed by Vasile Mircea Popa - Romania

UP.534. **If $a, b > 0$ then:**

$$\int_a^b\int_a^b\left(\frac{x}{x^4 + y^2} + \frac{y}{y^4 + x^2}\right)dxdy \le \ln^2\left(\frac{b}{a}\right)$$

Proposed by Daniel Sitaru - Romania

UP.535. If $x, y, z > 1; x \neq y \neq z \neq x$ and

$$\log_{\frac{y}{z}} x + \log_{\frac{z}{x}} y + \log_{\frac{x}{y}} z = 0$$

then:

$$\frac{\log_2 x}{\log_2^2(\frac{y}{z})} + \frac{\log_2 y}{\log_2^2(\frac{z}{x})} + \frac{\log_2 z}{\log_2^2(\frac{x}{y})} = 0$$

Proposed by Daniel Sitaru - Romania

UP.536. If $1 < a \leq b; m \geq 1$ then:

$$\frac{(b-1)^{m+1} - (a-1)^{m+1}}{m+1} + b - a \leq \frac{b^{m+1} - a^{m+1}}{m+1}$$

Proposed by Daniel Sitaru - Romania

UP.537. Find:

$$\Omega = \lim_{n\to\infty} \frac{1}{n \cdot 8^n} \binom{4n}{n} \binom{4n}{2n} \binom{3n}{n}^{-2}$$

Proposed by Daniel Sitaru - Romania

UP.538. Solve for real numbers:

$$\int_4^x \frac{t^2+1}{t^3+1} dt = 2(\sqrt{x} - 2)$$

Proposed by Daniel Sitaru - Romania

UP.539. If $0 < a \leq b$ then:

$$\int_a^b e^{x^2} dx \geq (b-a) \cdot \sqrt[3]{a^2 + ab + b^2}$$

Proposed by Daniel Sitaru - Romania

UP.540. If $a, b \in \mathbb{R}, a \leq b, f : [a, b] \to (0, \infty), f$ - continuous then:

$$3\int_a^b f(x)dx + \frac{1}{(b-a)^2}\left(\int_a^b \frac{1}{f(x)} dx\right)^3 \geq 4(b-a)$$

Proposed by Daniel Sitaru - Romania

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