4689

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Solver for real numbers:

$$\begin{cases} x, y, z > 0 \\ x + y + z = 3 \\ x^{x^2} \cdot y^{y^2} \cdot z^{z^2} = \frac{1}{(x^2)^{xz} \cdot (y^2)^{yz} \cdot (z^2)^{zy}} \end{cases}$$

Solution 1 by proposer.
Let be $f : (0, \infty) \to \mathbb{R}; f(x) = \ln x$.

$$f'(x) = \frac{1}{x}; f''(x) = -\frac{1}{x^2} < 0; f \text{ - concave}$$

By Jensen's inequality:

$$f\left(\lambda_1 \cdot \frac{1}{x} + \lambda_2 \cdot \frac{1}{y} + \lambda_3 \cdot \frac{1}{z}\right) \geq \lambda_1 f\left(\frac{1}{x}\right) + \lambda_2 f\left(\frac{1}{y}\right) + \lambda_3 f\left(\frac{1}{z}\right);$$

$$\lambda_1, \lambda_2, \lambda_3 > 0; \lambda_1 + \lambda_2 + \lambda_3 = 1$$

$$\text{Let be: } \lambda_1 = \frac{x^2 + 2xz}{(x + y + z)^2}; \lambda_2 = \frac{y^2 + 2yx}{(x + y + z)^2}; \lambda_3 = \frac{z^2 + 2zy}{(x + y + z)^2}$$

$$\lambda_1 + \lambda_2 + \lambda_3 = \frac{x^2 + 2xz + y^2 + 2yx + z^2 + 2zy}{(x + y + z)^2} = \frac{(x + y + z)^2}{(x + y + z)^2} = 1$$

$$\ln\left(\frac{x^2 + 2xz}{(x + y + z)^2} \cdot \frac{1}{x} + \frac{y^2 + 2yx}{(x + y + z)^2} \cdot \frac{1}{y} + \frac{z^2 + 2zy}{(x + y + z)^2} \cdot \frac{1}{z}\right) \geq$$

$$\geq \frac{x^2 + 2xz}{(x + y + z)^2} \ln \frac{1}{x} + \frac{y^2 + 2yx}{(x + y + z)^2} \ln \frac{1}{y} + \frac{z^2 + 2zy}{(x + y + z)^2} \ln \frac{1}{z}$$

$$\ln\left(\frac{x + 2z}{9} + \frac{y + 2x}{9} + \frac{z + 2y}{9}\right) \geq$$

$$\geq -\frac{x^2 + 2xz}{9} \cdot \ln x - \frac{y^2 + 2yx}{9} \cdot \ln y - \frac{z^2 + 2zy}{9} \cdot \ln z$$

$$\ln\left(\frac{3(x + y + z)}{9}\right) \geq -\frac{1}{9} \ln\left(x^{x^2 + 2xz} \cdot y^{y^2 + 2yx} \cdot z^{z^2 + 2zy}\right)$$

$$-9 \ln\left(\frac{3 \cdot 3}{9}\right) \leq \ln\left(x^{x^2} \cdot y^{y^2} \cdot z^{z^2} \cdot x^{2xz} \cdot y^{2yx} \cdot z^{2zy}\right)$$

$$-9 \ln 1 = 0 \leq \ln\left(x^{x^2} \cdot y^{y^2} \cdot z^{z^2} \cdot x^{2xz} \cdot y^{2yx} \cdot z^{2zy}\right)$$

$$\ln(x^{x^2} \cdot y^{y^2} \cdot z^{z^2} \cdot (x^2)^{xz} \cdot (y^2)^{yx} \cdot (z^2)^{zy}\Big) \geq \ln 1$$

$$x^{x^2} \cdot y^{y^2} \cdot z^{z^2} \cdot (x^2)^{xz} \cdot (y^2)^{yz} \cdot (z^2)^{zy} \geq 1$$

$$x^{x^2} \cdot y^{y^2} \cdot z^{z^2} \geq \frac{1}{(x^2)^{xz} \cdot (y^2)^{yz} \cdot (z^2)^{zy}}$$

Equality holds for $x = y = z = 1$. $\square$

Solution 2 by Editorial Board.
We show that the only solution is the obvious on: $(x, y, z) = (1, 1, 1)$.
Let $f(t) = \log t + \frac{1}{t} - 1$ for $t > 0$. Then $f'(t) = \frac{t-1}{t^2}$ is negative for $0 < t < 1$ and positive for $t > 1$, so f has a unique global maximum point (0,1). Suppose $x, y, z > 0$ with $x + y + z = 3$ and $(x, y, z) \neq (1, 1, 1)$. Then

$$\begin{aligned} 0 &= (x+y+z)^2 - 3(x+y+z) \\ &= (x^2+2xz)\Big(1-\frac{1}{x}\Big) + (y^2+2yx)\Big(1-\frac{1}{y}\Big) + (z^2+2zy)\Big(1-\frac{1}{z}\Big) \\ &< (x^2+2xz)\log x + (y^2+2yx)\log y + (z^2+2zy)\log z \\ &= \log(x^{x^2+2xz} \cdot y^{y^2+2yx} \cdot z^{z^2+2zy}) \end{aligned}$$

Hence,

$$x^{x^2+2xz} \cdot y^{y^2+2yx} \cdot z^{z^2+2zy} > 1, \quad \text{or} \quad x^{x^2} \cdot y^{y^2} \cdot z^{z^2} > \frac{1}{(x^2)^{xz} \cdot (y^2)^{yx} \cdot (z^2)^{zy}}$$

So (x, y, z) can't be a solution, completing the proof.

□

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