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Find $x, y > 0$ such that:

$$\frac{1}{(x+1)^8} + \frac{1}{(y+1)^8} = \frac{1}{8(xy+1)^4}$$

Solution 1 by proposers.

$$(xy+1)\left(\frac{x}{y}+1\right) = ((\sqrt{xy})^2 + 1^2)\left(\left(\sqrt{\frac{x}{y}}\right)^2 + 1^2\right) \overset{\text{CBS}}{\geq}$$

$$\geq \left(\sqrt{xy} \cdot \sqrt{\frac{x}{y}} + 1 \cdot 1\right)^2 = (x+1)^2$$

$$(1) \qquad \frac{1}{(x+1)^2} \geq \frac{1}{(xy+1)(\frac{x}{y}+1)} = \frac{y}{(x+y)(xy+1)}$$

Analogous:

$$(2) \qquad \frac{1}{(y+1)^2} \geq \frac{x}{(x+y)(xy+1)}$$

By adding $(1);(2)$:

$$(3) \qquad \frac{1}{(x+1)^2} + \frac{1}{(y+1)^2} \geq \frac{x+y}{(x+y)(1+xy)} = \frac{1}{xy+1}$$

$$\frac{1}{(x+1)^8} + \frac{1}{(y+1)^8} = \frac{\left(\frac{1}{(x+1)^2}\right)^4}{1^3} + \frac{\left(\frac{1}{(y+1)^2}\right)^4}{1^3} \geq$$

$$(4) \qquad \overset{\text{RADON}}{\geq} \frac{\left(\frac{1}{(x+1)^2} + \frac{1}{(y+1)^2}\right)^4}{(1+1)^3} \overset{(3)}{\geq} \frac{\left(\frac{1}{xy+1}\right)^4}{8} = \frac{1}{8(xy+1)^4}$$

Equality in (4) holds for $x = y = 1$ □

Solution 2 by Didier Pinchon - France and Titu Zvonaru - Romania.
For two real numbers a and b, the following inequality is well-known and easily verified

$$a^2 + b^2 \geq \frac{1}{2}(a+b)^2$$

with equality if and only if $a = b$. Therefore, choosing first $a = \frac{1}{(x+1)^4}, b = \frac{1}{(x+1)^4}$ and then $a = \frac{1}{(x+1)^2}, b = \frac{1}{(x+1)^2}$,

$$\frac{1}{(x+1)^8} + \frac{1}{(y+1)^8} \geq \frac{1}{2}\left(\frac{1}{(x+1)^4} + \frac{1}{(y+1)^4}\right)^2 \geq \frac{1}{8}\left(\frac{1}{(x+1)^2} + \frac{1}{(y+1)^2}\right)^4$$

with equality in each case if and only if $x = y$.
Now

$$\frac{1}{(x+1)^2} + \frac{1}{(y+1)^2} \geq \frac{1}{xy+1}$$

because

$$\frac{1}{(x+1)^2} + \frac{1}{(y+1)^2} - \frac{1}{xy+1} = \frac{xy(x-y)^2 + (xy-1)^2}{(x+1)^2(y+1)^2(xy+1)} \geq 0,$$

and, for $x, y > 0$, the equality holds if and only if $x = y$ and $xy = 1$, i.e. $x = 1$ and $y = 1$.
As a result,

$$\frac{1}{(x+1)^8} + \frac{1}{(y+1)^8} \geq \frac{1}{8(xy+1)^4},$$

with equality if and only if $x = 1$ and $y = 1$, which is the only solution for the proposed equation.

□

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