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DANIEL SITARU - ROMANIA

Find:

$$\Omega = \lim_{n\to\infty} \frac{1}{n^{n+1}} \sum_{k=2}^{n^n} [\log_k n];$$

$[*]$ - the greatest integer function.

Solution by proposer.

$$\sum_{k=2}^{n^n} [\log_k n] =$$

$$= \sum_{2\le k<n} [\log_n k] + \sum_{n\le k<n^2} [\log_n k] + \sum_{n^2\le k<n^3} [\log_n k] + \dots$$

$$\dots + \sum_{n^{n-1}\le k<n^n} [\log_n k] + [\log_n n^n] =$$

$$= (n-1)\cdot 0 + (n^2-n)\cdot 1 + (n^3-n^2)\cdot 2 + (n^4-n^3)\cdot 3 + \dots + (n^n - n^{n-1})(n-1) + n =$$

$$= -n - n^2 - n^3 - \dots - n^{n-1} + (n-1)n^n + n =$$

$$= -\frac{n(n^n-1)}{n-1} + (n-1)n^n + n$$

$$\Omega = \lim_{n\to\infty} \frac{1}{n^{n+1}} \sum_{k=2}^{n^n} [\log_n k] =$$

$$= \lim_{n\to\infty} \left(\frac{n^{n+1}-n^n}{n^{n+1}} + \frac{1}{n^n} - \frac{n^n-1}{n^n(n-1)} \right) =$$

$$= 1 + \frac{1}{\infty} - \frac{1}{\infty} = 1 + 0 - 0 = 1$$

□

Solution 2 by Henry Ricardo - New York - USA.

$$\lim_{n\to\infty} \frac{1}{n^{n+1}} \sum_{k=2}^{n^n} \log_n k$$

We have

$$\frac{1}{n^{n+1}} \sum_{k=2}^{n^n} \log_n k = \frac{1}{n^{n+1}\ln n} \sum_{k=2}^{n^n} \ln k = \frac{1}{n^{n+1}\ln n} \ln(n^n!)$$

Now we apply Stirling's approximation in the form

$$\ln n! = \frac{1}{2}\ln 2\pi + \frac{2n+1}{2}\ln n - n + O\left(\frac{1}{n}\right)$$

to obtain

$$\frac{\ln(n^n!)}{n^{n+1}\ln n} = \frac{\frac{1}{2}\ln 2\pi + (\frac{2n^n+1}{2})\cdot n\ln n - n^n + O(\frac{1}{n^n})}{n^{n+1}\ln n}$$

$$= 1 + \frac{\ln 2\pi}{2n^{n+1}\ln n} + \frac{1}{2n^n} - \frac{1}{n\ln n} + O\Big(\frac{1}{n^{2n+1}\ln n}\Big) \to 1 \text{ as } n \to \infty$$

□

Mathematics Department, National Economic College "Theodor Costescu", Drobeta Turnu - Severin, Romania

Email address: dansitaru63@yahoo.com