4819

DANIEL SITARU - ROMANIA

Let be $f:[0,1]\to[0,1], f$ - continuous function and $0<a\le b<1$. Prove that:

$$2\int_{\frac{2ab}{a+b}}^{\frac{a+b}{2}} tf(t)dt \ge \int_{\frac{2ab}{a+b}}^{\frac{a+b}{2}} f(t)dt\left(\int_0^{\frac{a+b}{2}} f(t)dt+\int_0^{\frac{2ab}{a+b}} f(t)dt\right)$$

Solution 1 by proposer.

$$\text{Let be } F:[0,1]\to\mathbb{R}, F(x)=\left(\int_0^x f(t)dt\right)^2-\int_0^x tf(t)dt$$

$$F'(x)=2\left(\int_0^x f(t)dt\right)'\left(\int_0^x f(t)dt\right)-2\left(\int_0^x tf(t)dt\right)'=$$

$$=2f(x)\int_0^x f(t)dt-2xf(x)=2f(x)\left(\int_0^x f(t)dt-x\right)=$$

$$=2f(x)\left(\int_0^x f(t)dt0-\int_0^x dt\right)=2f(x)\int_0^x (f(t)-1)dt\le 0,$$

because $f(x)\ge 0, f(x)\le 1, \forall x\in[0,1], F$ - decreasing

$$\frac{2ab}{a+b}\overset{\text{AM-HM}}{\le}\frac{a+b}{2}\Rightarrow F\left(\frac{2ab}{a+b}\right)\ge F\left(\frac{a+b}{2}\right)$$

$$\left(\int_0^{\frac{2ab}{a+b}} f(t)dt\right)^2-2\int_0^{\frac{2ab}{a+b}} tf(t)dt\ge\left(\int_0^{\frac{a+b}{2}} f(t)dt\right)^2-2\int_0^{\frac{a+b}{2}} tf(t)dt$$

$$2\int_{\frac{2ab}{a+b}}^{\frac{a+b}{2}} tf(t)dt\ge\left(\int_0^{\frac{a+b}{2}} f(t)dt-\int_0^{\frac{2ab}{a+b}} f(t)dt\right)\left(\int_0^{\frac{a+b}{2}} f(t)dt+\int_0^{\frac{2ab}{a+b}} f(t)dt\right)$$

$$2\int_{\frac{2ab}{a+b}}^{\frac{a+b}{2}} tf(t)dt\ge\int_{\frac{2ab}{a+b}}^{\frac{a+b}{2}} f(t)dt\left(\int_0^{\frac{a+b}{2}} f(t)dt+\int_0^{\frac{2ab}{a+b}} f(t)dt\right)$$

Equality holds for $a=b$ or $f\equiv 0$. □

Solution 2 by Editorial Board of Crux.

Let $x=2ab/(a+b)$ and $y=(a+b)/2$. Clearly $0<x\le y<1$. The desired inequality is equivalent to

$$2\left(\int_0^y tf(t)dt-\int_0^x tf(t)dt\right)\ge\left(\int_0^y f(t)dt-\int_0^x f(t)dt\right)\left(\int_0^y f(t)dt+\int_0^x f(t)dt\right)$$

Thus, it suffices to show that

$$\left(\int_0^x f(t)dt\right)^2 - 2\int_0^x tf(t)dt \geq \left(\int_0^y f(t)dt\right)^2 - 2\int_0^y tf(t)dt.$$

Let

$$g(z) = \left(\int_0^z f(t)dt\right)^2 - 2\int_0^z tf(t)dt,$$

with $z \in (0,1)$. Since f is continuous, the Fundamental Theorem of Calculus gives

$$g'(z) = 2f(z)\left(\int_0^z f(t)dt - z\right) \leq 0$$

since $f(t) \in [0,1]$. Thus, $g(x) \geq g(y)$, as required.

□

Mathematics Department, National Economic College "Theodor Costescu", Drobeta Turnu - Severin, Romania
Email address: dansitaru63@yahoo.com