A SIMPLE PROOF FOR BERNOULLI'S GENERALIZED INEQUALITY AND APPLICATIONS

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Abstract. In this paper we will give a simple proof for Bernoulli's generalized inequality and a few applications.

BERNOULLI'S GENERALIZED INEQUALITY

If $n \in \mathbb{N}^*; x_1, x_2, \ldots, x_n \in [-1, \infty); x_i x_j \geq 0; (\forall) i, j \in \overline{1, n}$ then:

$$(1+x_1)(1+x_2) \cdot \ldots \cdot (1+x_n) \geq 1 + x_1 + x_2 + \ldots + x_n \tag{1}$$

Proof.

We will use the mathematical induction:

For $n = 2$ we must prove that:

$$(1+x_1)(1+x_2) \geq 1 + x_1 + x_2$$

$$1 + x_1 + x_2 + x_1 x_2 \geq 1 + x_1 + x_2$$

$$x_1 x_2 \geq \text{ (True)}$$

$$P(n) : \prod_{i=1}^{n} (1 + x_i) \geq 1 + \sum_{i=1}^{n} x_i \text{ (suppose true)}$$

$$P(n+1) : \prod_{i=1}^{n+1} (1 + x_i) \geq 1 + \sum_{i=1}^{n+1} x_i \text{ (to prove)}$$

$$\prod_{i=1}^{n+1} (1 + x_i) = (1 + x_{n+1}) \cdot \prod_{i=1}^{n} (1 + x_i) \overset{P(n)}{\geq}$$

$$\geq (1 + x_{n+1})(1 + \sum_{i=1}^{n} x_i) =$$

$$= 1 + \sum_{i=1}^{n} x_i + x_{n+1} + x_{n+1} \sum_{i=1}^{n} x_i \geq 1 + \sum_{i=1}^{n+1} x_i$$

$$\sum_{i=1}^{n+1} x_i + \sum_{i=1}^{n} x_i \cdot x_{n+1} \geq \sum_{i=1}^{n+1} x_i$$

$$\sum_{i=1}^{n} x_i x_{n+1} \geq 0 \text{ (true)}$$

$$P(n) \to P(n+1)$$

□

Corollary 1.
If $a \in [-1, \infty); n \in \mathbb{N}^*$ then:

$$(1+a)^n \geq 1 + na \tag{2}$$

Proof.
We take in $x_1 = x_2 = \ldots = x_n = a$ □

Corollary 2.
If $x \geq 0; n \in \mathbb{N}^*$ then:

$$x^n \geq 1 + (n-1)x$$

Proof.
We take in $(2) : a = x - 1$. □

Application 1.
If $a_1, a_2, \ldots, a_n \in [\frac{1}{e}, \infty)$ then:

$$\ln(ea_1) \cdot \ln(ea_2) \cdot \ldots \cdot \ln(ea_n) \geq \ln(ea_1 a_2 \cdot \ldots \cdot a_n)$$

Proof.
We take in $(1) : x_1 = \ln a_1; x_2 = \ln a_2; \ldots; x_n = \ln a_n$

$$\prod_{i=1}^{n}(1 + \ln a_i) \geq 1 + \sum_{i=1}^{n} \ln a_i$$

$$\prod_{i=1}^{n}(\ln e + \ln a_i) \geq \ln e + \ln a_1 + \ln a_2 + \ldots + \ln a_n$$

$$\prod_{i=1}^{n} \ln(ea_i) \geq \ln(ea_1 a_2 \cdot \ldots \cdot a_n)$$

$$\ln(ea_1) \cdot \ln(ea_2) \cdot \ldots \cdot \ln(ea_n) \geq \ln(ea_1 a_2 \cdot \ldots \cdot a_n)$$

□

Application 2.
If $x, y, z \geq 0$ then:

$$e^{x+y+z} + 2 \geq e^x + e^y + e^z$$

Proof.
For $n = 3$ in (1):

$$(1 + x_1)(1 + x_2)(1 + x_3) \geq 1 + x_1 + x_2 + x_3$$

$$\text{Replace: } x_1 = e^x - 1; x_2 = e^y - 1; x_3 = e^z - 1$$

$$(1 + e^x - 1)(1 + e^y - 1)(1 + e^z - 1) \geq 1 + e^x - 1 + e^y + e^z - 1$$

$$e^x \cdot e^y \cdot e^z \geq e^x + e^y + e^z - 2$$

$$e^{x+y+z} + 2 \geq e^x + e^y + e^z$$

Equality holds for $x = y = z = 0$. □

References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

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