A SIMPLE PROOF FOR MADEVSKI'S INEQUALITY AND APPLICATIONS

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ABSTRACT. In this paper we will give a simple proof for Madevski's inequality and a few applications.

MADEVSKI'S INEQUALITY

If $x > 1; p \geq 1$ then:

$$(x-1)^p \leq x^p - 1 \tag{1}$$

Proof.

$$\text{Let be } f : (1, \infty) \to \mathbb{R}; f(x) = (x-1)^p - x^p + 1$$
$$f'(x) = p(x-1)^{p-1} - px^{p-1} = p((x-1)^{p-1} - x^{p-1}) < 0$$
$$\text{because } x - 1 < x; (\forall) x > 1.$$
$$f \text{ decreasing on } (1, \infty)$$
$$\sup_{x>1} f(x) = \lim_{\substack{x \to 1 \\ x>1}} f(x) = \lim_{\substack{x \to 1 \\ x>1}} ((x-1)^p - x^p + 1) =$$
$$= (1-1)^p - 1^p + 1 = -1 + 1 = 0$$
$$\Rightarrow f(x) \leq 0; (\forall) x > 1$$
$$(x-1)^p - x^p + 1 \leq 0$$
$$(x-1)^p \leq x^p - 1$$

Equality holds for $p = 1$. □

Corollary 1.

If $x > 1, m, n \geq 1$ then:

$$(x-1)^{m+n} + x^m + x^n \leq x^{m+n} + 1 \tag{2}$$

Proof.

By (1):

$$(x-1)^m \leq x^m - 1 \tag{3}$$

$$(x-1)^n \leq x^n - 1 \tag{4}$$

By multiplying (3); (4):

$$(x-1)^m \cdot (x-1)^n \leq (x^m - 1)(x^n - 1)$$
$$(x-1)^{m+n} \leq x^{m+n} - x^m - x^n + 1$$
$$(x-1)^{m+n} + x^m + x^n \leq x^{m+n} + 1$$

Equality holds for $m = n = 1$. □

Corollary 2.
If $x > 1; m, n, p \geq 1$ then:

$$(x-1)^{m+n+p} + x^m + x^n + x^p \leq x^{m+n} + x^{n+p} + x^{p+m} + 1 \tag{5}$$

Proof.
By (1):

$$(x-1)^p \leq x^p - 1 \tag{6}$$

By multiplying $(3); (4); (6)$:

$$(x-1)^m \cdot (x-1)^n \cdot (x-1)^p \leq (x^m - 1)(x^n - 1)(x^p - 1)$$
$$(x-1)^{m+n+p} \leq (x^{m+n} - x^m - x^n + 1)(x^p - 1)$$
$$(x-1)^{m+n+p} \leq x^{m+n+p} - x^{m+n} - x^{m+p} + x^m - x^{n+p} + x^n + x^p - 1$$
$$(x-1)^{m+n+p} + x^{m+n} + x^{m+p} + x^{n+p} \leq x^{m+n+p} + x^m + x^n + x^p + 1$$

Equality holds for: $m = n = p = 1$ □

Corollary 3.
If $a \in \mathbb{R}; m, n \geq 1$ then:

$$(2 + \sin a)^m + 1 \leq (3 + \sin a)^m \tag{7}$$

$$(2 + \sin a)^{m+n} + (3 + \sin a)^m + (3 + \sin a)^n \leq (3 + \sin a)^{m+n} + 1 \tag{8}$$

Proof.
We take in $(1) : x = 3 + \sin a > 1$ and we obtain (7).
We take in $(2) : x = 3 + \sin a > 1$ and we obtain (8). □

Corollary 4.
If $m, n, p \geq 1$ then:

$$(e-1)^{m+1} + e^m + e^n \leq e^{m+n} + 1 \tag{9}$$

$$(e-1)^{m+n+p} + e^{m+n} + e^{n+p} + e^{p+m} \leq e^{m+n+p} + e^m + e^n + e^p + 1 \tag{10}$$

Proof.
We take in $(2) : x = e > 1$ and we obtain (9).
We take in $(5) : x = e > 1$ and we obtain (10). □

References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

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