A SIMPLE PROOF FOR OSTROWSKI'S INEQUALITY AND APPLICATIONS

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ABSTRACT. In this paper we will give a simple proof for Ostrowski's inequality and a few applications.

OSTROWSKI'S INEQUALITY

If $0 \le x \le 1; n \in \mathbb{N}; n \ge 1$ then:

$$0 \le x^{n-1} \cdot (1-x)^n \le \frac{1}{2^{2(n-1)}} \tag{1}$$

Proof.

$$x \ge 0 \Rightarrow x^{n-1} \ge 0 \tag{2}$$

$$x \le 1 \Rightarrow -x \le -1 \Rightarrow 1-x \ge 0 \Rightarrow (1-x)^n \ge 0 \tag{3}$$

By $(2); (3)$:

$$x^{n-1} \cdot (1-x)^n \ge 0$$

$$x^{n-1} \cdot (1-x)^n = (1-x) \cdot x^{n-1} \cdot (1-x)^{n-1} =$$

$$= (1-x)(x(1-x))^{n-1} \overset{\text{AM-GM}}{\le}$$

$$\le (1-x) \cdot \left(\left(\frac{x+1-x}{2} \right)^2 \right)^{n-1} =$$

$$= (1-x) \cdot \left(\left(\frac{1}{2} \right)^2 \right)^{n-1} = (1-x) \cdot \frac{1}{2^{2(n-1)}} \le \frac{1}{2^{2(n-1)}}$$

Equality holds for: $x = 1 - x \Rightarrow x = \frac{1}{2}$ □

Corollary 1.

If $a \in \mathbb{R}; n \in \mathbb{N}; n \ge 1$ then:

$$0 \le \sin^{2n-2} a \cdot \cos^{2n} a \le \frac{1}{2^{2(n-1)}}$$

Proof.

We take in $(1) : x = \sin^2 a$

$$0 \le (\sin^2 a)^{n-1} \cdot (1 - \sin^2 a)^n \le \frac{1}{2^{2(n-1)}}$$

$$0 \le \sin^{2n-2} a \cdot \cos^{2n} a \le \frac{1}{2^{2(n-1)}}$$

Equality holds for:

$$\sin^2 a = \cos^2 a \Rightarrow \tan^2 a = 1 \Rightarrow \tan a = \pm 1$$

$$x \in \left\{ \frac{\pi}{4} + k\pi \middle| k \in \mathbb{Z} \right\} \cup \left\{ \frac{3\pi}{4} + k\pi \middle| k \in \mathbb{Z} \right\}$$

□

Corollary 2.
If $a \in \mathbb{R}; n \in \mathbb{N}; n \geq 1$ then:

$$0 \leq \cos^{2n-2} a \cdot \sin^{2n} a \leq \frac{1}{2^{2(n-1)}}$$

Proof.
We take in $(1) : x = \cos^2 a$

$$0 \leq (\cos^2 a)^{n-1} \cdot (1 - \cos^2 a)^n \leq \frac{1}{2^{2(n-1)}}$$

$$0 \leq \cos^{2n-2} a \cdot \sin^{2n} a \leq \frac{1}{2^{2(n-1)}}$$

Equality holds for:

$$\sin^2 a = \cos^2 a \Rightarrow \tan^2 a = 1 \Rightarrow \tan a = \pm 1$$

$$x \in \left\{ \frac{\pi}{4} + k\pi \middle| k \in \mathbb{Z} \right\} \cup \left\{ \frac{3\pi}{4} + k\pi \middle| k \in \mathbb{Z} \right\}$$

□

Corollary 3.
If $a \in \mathbb{R}; n \in \mathbb{N}; n \geq 1$ then:

$$0 \leq \frac{(1 - e^{-a})^n}{e^{(n-1)a}} \leq \frac{1}{2^{2(n-1)}}$$

Proof.
We take in $(1) : x = e^{-a}$

$$0 \leq (e^{-a})^{n-1} \cdot (1 - e^{-a})^n \leq \frac{1}{2^{2(n-1)}}$$

$$0 \leq \frac{(1 - e^{-a})^n}{e^{(n-1)a}} \leq \frac{1}{2^{2(n-1)}}$$

Equality holds for:

$$e^{-a} = 1 - e^{-a} \Rightarrow e^{-a} = \frac{1}{2} \Rightarrow e^a = 2 \Rightarrow a = \ln 2$$

□

Corollary 4.
If $a \in [1, e]; n \in \mathbb{N}; n \geq 1$ then:

$$0 \leq \ln^{n-1} a \cdot (1 - \ln a)^n \leq \frac{1}{2^{2(n-1)}}$$

Proof.

$$a \in [1, e] \Rightarrow 1 \leq a \leq e \Rightarrow 0 \leq \ln a \leq 1$$

We take in $(1) : x = \ln a$

$$0 \leq \ln^{n-1} a \cdot (1 - \ln a)^n \leq \frac{1}{2^{2(n-1)}}$$

Equality holds for:

$$\ln a = 1 - \ln a \Rightarrow \ln a = \frac{1}{2} \Rightarrow a = \sqrt{e}$$

□

References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

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