B131

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Evaluate

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(m-n)^2}{mn(m+1)^2(n+1)^2(m+2)(n+2)}$$

Solution 1 by proposer.

$$\sum_{i=1}^{n}\frac{1}{i(i+1)}=\sum_{i=1}^{n}\Big(\frac{1}{i}-\frac{1}{i+1}\Big)=1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\ldots+\frac{1}{n}-\frac{1}{n+1}=1-\frac{1}{n+1}$$

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=1$$

$$\sum_{i=1}^{n}\frac{1}{(i+1)(i+2)}=\sum_{i=1}^{n}\Big(\frac{1}{i+1}-\frac{1}{i+2}\Big)=\frac{1}{2}-\frac{1}{3}=\frac{1}{3}-\frac{1}{4}+\ldots+\frac{1}{n+1}-\frac{1}{n+2}$$

$$=\frac{1}{2}-\frac{1}{n+2}$$

$$\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)}=\frac{1}{2}$$

$$\sum_{i=1}^{n}\frac{1}{i(i+2)}=\frac{1}{2}\sum_{i=1}^{n}\frac{2}{i(i+2)}=\frac{1}{2}\sum_{i=1}^{n}\frac{(i+2)-i}{i(i+2)}=\frac{1}{2}\sum_{i=1}^{n}\Big(\frac{1}{i}-\frac{1}{i+2}\Big)=$$

$$=\frac{1}{2}\Big(1-\frac{1}{3}+\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{5}+\ldots+\frac{1}{n-2}-\frac{1}{n}+\frac{1}{n-1}-\frac{1}{n+1}+\frac{1}{n}-\frac{1}{n+2}\Big)=$$

$$=\frac{1}{2}\Big(1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}\Big)$$

$$\sum_{n=1}^{\infty}\frac{1}{n(n+2)}=\frac{1}{2}\Big(1+\frac{1}{2}\Big)=\frac{3}{4}$$

$$\sum_{n=1}^{\infty}\frac{1}{(n+1)^2}=\lim_{n\to\infty}\Big(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{(n+1)^2}-\frac{1}{1^2}\Big)=\frac{\pi^2}{6}-1$$

$$\Omega=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(m-n)^2}{mn(m+1)^2(n+1)^2(m+2)(n+2)}=$$

$$=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{m-n}{mn(m+1)(n+1)}\cdot\frac{m-n}{(m+1)(m+2)(n+1)(n+2)}=$$

$$=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{m(n+1)-n(m+1)}{mn(m+1)(n+1)}\cdot\frac{(m+1)(n+2)-(m+2)(n+1)}{(m+1)(m+2)(n+1)(n+2)}=$$

$$=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Big(\frac{1}{n(m+1)}-\frac{1}{m(n+1)}\Big)\Big(\frac{1}{(n+1)(m+2)}-\frac{1}{(m+1)(n+2)}\Big)\overset{\text{Binet-Cauchy}}{=}$$

$$=2\left(\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\right)\left(\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)}\right)-2\left(\sum_{n=1}^{\infty}\frac{1}{n(n+2)}\right)\left(\sum_{n=1}^{\infty}\frac{1}{(n+1)^2}\right)=$$

$$=2\cdot 1\cdot\frac{1}{2}-2\cdot\frac{3}{4}\Big(\frac{\pi^2}{6}-1\Big)=1-\frac{3}{2}\cdot\frac{\pi^2}{6}+\frac{3}{2}=\frac{5}{2}-\frac{\pi^2}{4}=\frac{10-\pi^2}{4}$$

Observation.
Binet-Cauchy's identity:

If $a_i, b_i, x_i, y_i > 0, i=\overline{1,n}$ then:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(a_ib_j-a_jb_i)(x_iy_j-x_jy_i)=$$

$$=2\left(\sum_{i=1}^{n}a_ix_i\right)\left(\sum_{i=1}^{n}b_iy_i\right)-2\left(\sum_{i=1}^{n}a_iy_i\right)\left(\sum_{i=1}^{n}b_ix_i\right)$$

For $a_i=\frac{1}{i}, b_i=\frac{1}{i+1}, x_i=\frac{1}{i+1}, y_i=\frac{1}{i+2}$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\Big(\frac{1}{i(j+1)}-\frac{1}{j(i+1)}\Big)\Big(\frac{1}{(i+1)(j+2)}-\frac{1}{(j+1)(i+2)}\Big)=$$

$$=2\left(\sum_{i=1}^{n}\frac{1}{i(i+1)}\right)\left(\sum_{i=1}^{n}\frac{1}{(i+1)(i+2)}\right)-2\left(\sum_{i=1}^{n}\frac{1}{i(i+2)}\right)\left(\sum_{i=1}^{n}\frac{1}{(i+2)^2}\right)$$

□

Solution 2 by Rana Ranino - Setif - Algerie.

$$\Omega=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(m-n)^2}{mn(m+1)^2(n+1)^2(m+2)(n+2)}$$

$$\Omega=\sum_{n=1}^{\infty}\frac{1}{n(n+1)^2(n+2)}\sum_{m=1}^{\infty}\frac{m}{(m+1)^2(m+2)}+$$

$$+\sum_{n=1}^{\infty}\frac{n}{(n+1)^2(n+2)}\sum_{m=1}^{\infty}\frac{1}{m(m+1)^2(m+2)}-$$

$$-2\sum_{n=1}^{\infty}\frac{1}{(n+1)^2(n+2)}\sum_{m=1}^{\infty}\frac{1}{(m+1)^2(m+2)}$$

By Symmetry: $\Omega=2\sum_{n=1}^{\infty}\frac{1}{n(n+1)^2(n+2)}\sum_{m=1}^{\infty}\frac{m}{(m+1)^2(m+2)}-$

$$-2\Big(\sum_{n=1}^{\infty}\frac{1}{(n+1)^2(n+2)}\Big)^2$$

$$\Omega_1=\sum_{m=1}^{\infty}\frac{m}{(m+1)^2(m+2)}=2\sum_{m=1}^{\infty}\Big(\frac{1}{m+1}-\frac{1}{m+2}\Big)-$$

$$-\sum_{m=1}^{\infty}\frac{1}{(m+1)^2}=2-\frac{\pi^2}{6}$$

$$\Omega_2=\sum_{n=1}^{\infty}\frac{2}{n(n+1)^2(n+2)}=\sum_{n=1}^{\infty}\Big(\frac{1}{n}-\frac{1}{n+2}\Big)-$$

$$-2\sum_{m=1}^{\infty}\frac{1}{(n+1)^2}=\frac{3}{2}+2-\frac{\pi^2}{3}=\frac{7}{2}-\frac{\pi^2}{3}$$

$$\Omega_3=\sum_{n=1}^{\infty}\frac{1}{(n+1)^2(n+2)}=\sum_{n=1}^{\infty}\Big(\frac{1}{n+2}-\frac{1}{n+1}\Big)+$$

$$+\sum_{m=1}^{\infty}\frac{1}{(n+1)^2}=-\frac{1}{2}+\frac{\pi^2}{6}-1=\frac{\pi^2}{6}-\frac{3}{2}$$

$$\Omega=\Big(2-\frac{\pi^2}{6}\Big)\Big(\frac{7}{2}-\frac{\pi^2}{3}\Big)-2\Big(\frac{\pi^2}{6}-\frac{3}{2}\Big)^2=\frac{5}{2}-\frac{\pi^2}{4}$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(m-n)^2}{mn(m+1)^2(n+1)^2(m+2)(n+2)}=\frac{5}{2}-\frac{\pi^2}{4}$$

□

Solution 3 by Ankush Kumar Parcha - India.

$$\text{We have }\sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}}\frac{(n-m)^2}{nm(n+1)^2(m+1)^2(n+2)(m+2)}$$

$$\overset{\therefore\text{ symmetry}}{\Longrightarrow} 2\sum_{n,m\in\mathbb{N}}\frac{n}{m(n+1)^2(m+1)^2(n+2)(m+2)}-2\left(\sum_{n\in\mathbb{N}}\frac{1}{(n+1)^2(n+2)}\right)^2$$

$$\underset{(1)}{\overset{\text{Note section}}{\Longrightarrow}} 2\int_0^{\infty}\frac{1}{e^x-1}\mathcal{L}_n^{-1}\Big\{\frac{n}{(n+1)^2(n+2)}\Big\}(x)dx\cdot$$

$$\cdot\int_0^{\infty}\frac{1}{e^y-1}\mathcal{L}_m^{-1}\Big\{\frac{1}{m(m+1)^2(m+2)}\Big\}(y)dy-$$

$$-2\left(\int_0^{\infty}\frac{1}{e^z-1}\mathcal{L}_n^{-1}\Big\{\frac{1}{(n+1)^2(n+2)}\Big\}(z)dx\right)^2$$

$$\underset{(2)}{\overset{\text{Note section}}{\Longrightarrow}} 2\int_0^{\infty}\frac{xe^{-2x}+2e^{-3x}-2e^{-2x}}{1-e^{-x}}dx\int_0^{\infty}\frac{1}{1-e^{-y}}\Big(ye^{-2y}+\frac{e^{-3y}}{2}-\frac{e^{-y}}{2}\Big)dy-$$

$$-2\left(\int_0^{\infty}\frac{ze^{-2z}+e^{-3z}-e^{-2z}}{1-e^{-z}}dx\right)^2$$

$$\underset{(3)}{\overset{\text{Note section}}{\Longrightarrow}} 2\left(\zeta(2,2)-2\int_0^{\infty}e^{-2x}dx\right)\left(\zeta(2,2)+\int_0^{\infty}d\Big(\frac{e^{-2y}}{4}+\frac{e^{-y}}{2}\Big)\right)$$

$$-2\left(\zeta(2,2)-\int_0^{\infty}e^{-2z}dx\right)\Rightarrow 2(\zeta(2)-2)\Big(\zeta(2)-\frac{7}{4}\Big)-2\Big(\zeta(2)-\frac{3}{2}\Big)^2$$

$$\overset{\therefore\zeta(2)=\pi^2/6}{\Longrightarrow}\frac{\pi^4}{18}-\frac{5\pi^2}{4}+7-\frac{\pi^4}{18}+\pi^2-\frac{9}{2}$$

$$\Rightarrow\sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}}\frac{(n-m)^2}{nm(n+1)^2(m+1)^2(n+2)(m+2)}=\frac{5}{2}-\frac{\pi^2}{4}$$

Note Section

1. $\displaystyle\sum_{n\in\mathbb{N}}\mathcal{L}_t\{f(t)\}(n)=\int_0^{\infty}\frac{f(t)}{e^t-1}dt$ (Maz. sum identity)

$$2.\ \mathcal{L}_x\{x^n e^{ax}\}(s) = \frac{n!}{(s-a)^{n+1}}, n \in \mathbb{N} \wedge \Re(s) > \Re(a)$$

$$3.\ \int_0^\infty \frac{t^{n-1}e^{-mt}}{1-e^{-t}} dt = \Gamma(n)\cdot\zeta(n,m), \Re(n) > 1 \wedge \Re(m) > 0$$

$\square$

Solution 4 by Ravi Prakash - New Delhi - India.

$$\text{Let } a_{m,n} = \frac{(m-n)^2}{mn(m+1)^2(n+1)^2(m+2)(n+2)}$$

$$= \frac{(m+1)^2+(n+1)^2-2(m+1)(n+1)}{mn(m+1)^2(n+1)^2(m+2)(n+2)}$$

$$= \frac{1}{mn(n+1)^2(m+2)(n+2)} + \frac{1}{mn(m+1)^2(m+2)(n+2)} -$$

$$-\frac{2}{(mn(m+1)(n+1)(m+2)(n+2)}$$

$$S = \sum_{m=1}^\infty \sum_{n=1}^\infty a_{mn}$$

$$= 2\sum_{m=1}^\infty \frac{1}{m(m+2)} \sum_{n=1}^\infty \frac{1}{n(n+1)^2(n+2)} - 2\left[\sum_{m=1}^n \frac{1}{m(m+1)(m+2)}\right]^2$$

$$= \left[\sum_{m=1}^\infty \left(\frac{1}{m} - \frac{1}{m+2}\right)\right]\left[\sum_{n=1}^\infty \frac{1}{n(n+1)^2(n+2)}\right] - 2\left[\sum_{m=1}^\infty \frac{1}{m(m+1)(m+2)}\right]^2$$

$$= \frac{3}{2}A_1 - 2A_2^2$$

where

$$A_1 = \sum_{n=1}^\infty \frac{1}{n(n+1)^2(n+2)} = \sum_{n=1}^\infty \frac{1}{(n+1)^2[(n+1)^2-1]}$$

$$= \sum_{n=1}^\infty \left[\frac{1}{(n+1)^2-1} - \frac{1}{(n+1)^2}\right]$$

$$= \sum_{n=1}^\infty \left[\frac{1}{2n} - \frac{1}{2(n+2)} - \frac{1}{(n+1)^2}\right]$$

$$= \frac{1}{2}\left(1+\frac{1}{2}\right) - \sum_{n=1}^\infty \frac{1}{(n+1)^2}$$

$$= \frac{3}{4} - \left(\frac{\pi^2}{6} - 1\right) = \frac{7}{4} - \frac{\pi^2}{6}$$

Next,

$$A_2 = \sum_{n=1}^\infty \frac{1}{n(n+1)(n+2)} = \frac{1}{2}\sum_{n=1}^\infty \left(\frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}\right)$$

$$= \frac{1}{2}\left[\sum_{n=1}^\infty \left(\frac{1}{n} - \frac{1}{n+1}\right) - \sum_{n=1}^\infty \left(\frac{1}{n+1} - \frac{1}{n+2}\right)\right]$$

$$= \frac{1}{2}\left[1 - \frac{1}{2}\right] = \frac{1}{4}$$

Thus,

$$S = \frac{3}{2}\Big(\frac{7}{4} - \frac{\pi^2}{6}\Big) - 2\Big(\frac{1}{4}\Big)^2$$
$$= \frac{21}{8} - \frac{1}{8} - \frac{\pi^2}{4} = \frac{5}{2} - \frac{\pi^2}{4}$$

□

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