B135

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If $A, B, C \in M_2(\mathbb{R}); \det A > 0; \det B > 0; \det C > 0, \det(ABC) = 64$ then:

$$\det(A+B+C) + \det(-A+B+C) + \det(A-B+C) + \det(A+B-C) \geq 48$$

Solution 1 by proposer.
Lemma: If $A, B \in M_2(\mathbb{R})$ then:

$$\det(A+B) + \det(A-B) = 2(\det A + \det B)$$

Proof.

$$\text{Let be } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

$$\det(A+B) + \det(A-B) =$$
$$= \begin{vmatrix} a+e & b+f \\ c+g & d+h \end{vmatrix} + \begin{vmatrix} a-e & b-f \\ c-g & d-h \end{vmatrix} =$$
$$= (a+e)(d+h) - (b+f)(c+g) + (a-e)(d-h) - (c-g)(b-f) =$$
$$= ad + ah + de + eh - bc - bg - cf - fg +$$
$$+ ad - ah - de + eh - cb + cf + bg - gf =$$
$$= 2ad - 2bc + 2eh - 2gf =$$
$$= 2(ad - bc) + 2(eh - gf) =$$
$$= 2\det A + 2\det B = 2(\det A + \det B)$$

□

Back to the problem:

$$\det(A+B+C) + \det(-A+B+C) + \det(A-B+C) + \det(A+B-C) =$$
$$= (\det((A+B)+C) + \det((A+B)-C) +$$
$$+ \det(C+(A-B)) + \det(C-(A-B))) \overset{\text{Lemma}}{=}$$
$$= 2(\det(A+B) + \det C) + 2(\det(C) + \det(A-B)) =$$
$$= 4\det C + 2(\det(A+B) + \det(A-B)) =$$
$$\overset{\text{Lemma}}{=} 4\det C + 2 \cdot 2(\det A + \det B) =$$
$$= 4(\det A + \det B + \det C) \overset{\text{AM-GM}}{\geq}$$
$$\geq 4 \cdot 3\sqrt[3]{\det A \cdot \det B \cdot \det C} =$$
$$= 12 \cdot \sqrt[3]{\det(ABC)} = 12 \cdot \sqrt[3]{64} = 12 \cdot 4 = 48$$

Equality holds for:

$$\det A = \det B = \det C = 4$$

□

Solution 2 by Ravi Prakash - New Delhi - India.

$$\text{Let } A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \text{ and } C = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$$

$$\det(A+B) = \begin{vmatrix} a1 + b_1 & a_2 + b_2 \\ a_3 + b_3 & a_4 + b_4 \end{vmatrix} =$$

$$= \begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix} + \begin{vmatrix} a_1 & b_2 \\ a_3 & b_4 \end{vmatrix} + \begin{vmatrix} b_1 & a_2 \\ b_3 & a_4 \end{vmatrix} + \begin{vmatrix} b_1 & b_2 \\ b_3 & b_4 \end{vmatrix}$$

$$= \det(A_1|A_2) + \det(A_1|B_2) + \det(B_1|A_2) + \det(B_1|B_2)$$

$$\text{where } A_1 = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, A_2 = \begin{bmatrix} a_3 \\ a_4 \end{bmatrix}, \text{ etc}$$

$$\text{and } \det(A_1|A_2) = \begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix}, \text{ etc}$$

Similarly,

$$\det(A-B) = \det(A_1|A_2) - \det(A_1|B_2) - \det(B_1|A_2) + \det(B_1|B_2)$$

Thus,

$$\det(A+B) + \det(A-B) = 2\det(A_1|A_2) + 2\det(B_1|B_2)$$
$$= 2\det(A) + 2\det(B)$$

Thus,

$$\det(A+B+C) + \det(-A+B+C) + \det(A-B+C) + \det(A+B-C) =$$
$$= 2\det(B+C) + 2\det(A) + 2\det(B-C) + 2\det(A)$$
$$= 2[2\det(B) + 2\det(C)] + 4\det(A) =$$
$$= 4[\det(A) + \det(B) + \det(C)]$$
$$\geq (4)(3)[(\det(A))(\det(B))(\det(C))]^{\frac{1}{3}}$$
$$= 12[\det(ABC)]^{\frac{1}{3}} = (12)(4^3)^{\frac{1}{3}}$$
$$= 48$$

□

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