CLOSED FORM FOR SERIES INVOLVING FIBONACCI'S NUMBERS

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Abstract:
In this paper we present some infinite series involving fibonacci's numbers.

Propositions:

a). If

$$\Omega(\kappa,\lambda) := \sum_{n=0}^{\infty}\binom{2n}{n}\frac{F_{\kappa n}}{\lambda^n}$$

Then,

$$\Omega(\kappa,\lambda) = \sqrt{\frac{\lambda}{5}}\left[\frac{1}{\sqrt{\lambda-4\phi^{\kappa}}} - \frac{1}{\sqrt{\lambda-4\psi^{\kappa}}}\right]$$

where F_n- nth fibonacci number, $\kappa \in \mathbb{N},\ \phi = \dfrac{1+\sqrt{5}}{2}, \psi = \dfrac{1-\sqrt{5}}{2}$ and $\lambda \in \mathbb{R}^+ \ni \lambda > \max\{4\phi^k, 4\psi^k\}$

b).

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{F_{n+1}^2 - F_{n-1}^2}{2^{4n}} = \frac{1}{5}\sqrt{\frac{5}{2} - \sqrt{5}}$$

c).

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2F_n^3 + 3F_nF_{n+1}F_{n-1}}{2^{5n}} = \sqrt{\frac{6-\sqrt{31}}{155}}$$

PROOFS:

a). From the definition above,

$$\Omega(\kappa,\lambda) := \sum_{n=0}^{\infty}\binom{2n}{n}\frac{F_{\kappa n}}{\lambda^n}$$

Thus,

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{F_{\kappa n}}{\lambda^n} = \sum_{n=0}^{\infty}\binom{2n}{n}\frac{(1+\sqrt{5})^{\kappa n} - (1-\sqrt{5})^{\kappa n}}{2^{\kappa n}\sqrt{5}\lambda^n}$$

$$= \frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}\binom{2n}{n}\left[\frac{1}{\lambda}\left(\frac{1+\sqrt{5}}{2}\right)^{\kappa}\right]^n - \frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}\binom{2n}{n}\left[\frac{1}{\lambda}\left(\frac{1-\sqrt{5}}{2}\right)^{\kappa}\right]^n$$

$$= \frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}\binom{2n}{n}\left[\frac{1}{\lambda}(\phi)^{\kappa}\right]^n - \frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}\binom{2n}{n}\left[\frac{1}{\lambda}(\psi)^{\kappa}\right]^n$$

Recall the ordinary generating function for the central binomial coefficients is given by,

$$\frac{1}{\sqrt{1-4x}} = \sum_{n=0}^{\infty}\binom{2n}{n}x^n = 1 + 2x + 6x^2 + 20x^3 + 70x^4 + \cdots$$

It follows directly,

$$\frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}\binom{2n}{n}\left[\frac{1}{\lambda}(\phi)^{\kappa}\right]^n - \frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}\binom{2n}{n}\left[\frac{1}{\lambda}(\psi)^{\kappa}\right]^n = \frac{1}{\sqrt{5}}\left[\frac{\sqrt{\lambda}}{\sqrt{\lambda-4\phi^{\kappa}}} - \frac{\sqrt{\lambda}}{\sqrt{\lambda-4\psi^{\kappa}}}\right]$$

Which shows,

$$\Omega(\kappa,\lambda)=\sqrt{\frac{\lambda}{5}}\left[\frac{1}{\sqrt{\lambda-4\phi^{\kappa}}}-\frac{1}{\sqrt{\lambda-4\psi^{\kappa}}}\right]$$

EXAMPLE:
We have,

$$\Omega(1,2^3)=\sum_{n=0}^{\infty}\binom{2n}{n}\frac{F_n}{2^{3n}}=\frac{2\sqrt{2}}{\sqrt{5}}\left[\frac{1}{\sqrt{8-4\phi}}-\frac{1}{\sqrt{8-4\psi}}\right]=\sqrt{\frac{2}{5}}$$

b). Recall d'Ocagne's Identity,

$$F_{n+1}^2-F_{n-1}^2=F_{2n}$$

Then,

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{F_{n+1}^2-F_{n-1}^2}{2^{4n}}=\sum_{n=0}^{\infty}\binom{2n}{n}\frac{F_{2n}}{2^{4n}}=\Omega(2,2^4)$$

But

$$\Omega(2,2^4)=\frac{\sqrt{2}}{\sqrt{5}}\left[\frac{1}{\sqrt{16-4\phi^2}}-\frac{1}{\sqrt{16-4\psi^2}}\right]=\frac{1}{10}\left(\sqrt{5+\sqrt{5}}-\sqrt{5-\sqrt{5}}\right)=\frac{1}{5}\sqrt{\frac{5}{2}-\sqrt{5}}$$

Thus we conclude,

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{F_{n+1}^2-F_{n-1}^2}{2^{4n}}=\frac{1}{5}\sqrt{\frac{5}{2}-\sqrt{5}}$$

c). It is popularly known that,

$$F_{3n}=2F_n^3+3F_nF_{n+1}F_{n-1}$$

Then,

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2F_n^3+3F_nF_{n+1}F_{n-1}}{2^{5n}}=\sum_{n=0}^{\infty}\binom{2n}{n}\frac{F_{3n}}{2^{5n}}=\Omega(3,2^5)$$

But

$$\Omega(3,2^5)=\frac{\sqrt{2}}{\sqrt{5}}\left[\frac{1}{\sqrt{32-4\phi^3}}-\frac{1}{\sqrt{32-4\psi^3}}\right]=\frac{\sqrt{6+\sqrt{5}}-\sqrt{6-\sqrt{5}}}{\sqrt{310}}=\sqrt{\frac{6-\sqrt{31}}{155}}$$

Similarly we conclude,

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2F_n^3+3F_nF_{n+1}F_{n-1}}{2^{5n}}=\sqrt{\frac{6-\sqrt{31}}{155}}$$

REFERENCES
1. Abramowitz, M., and Stegun,I, A, (1972). Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables,Dover Publications, New-York.
2. Romanian Mathematical Magazine (RMM).