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JP.526 Let $a,b,c\in(0,1)$. Prove that:

$$\frac{b+c}{1-a}+\frac{c+a}{1-b}+\frac{a+b}{1-c}\geq\frac{2\sqrt{ab}}{1-\sqrt{ab}}+\frac{2\sqrt{bc}}{1-\sqrt{bc}}+\frac{2\sqrt{ca}}{1-\sqrt{ca}}$$

Proposed by Marin Chirciu – Romania

Solution 1 by proposer

We prove: Lemma: Let $a,b\in(0,1)$. Prove that:

$$\frac{a}{1-b}+\frac{b}{1-a}\geq\frac{2\sqrt{ab}}{1-\sqrt{ab}}$$

Solution: Denoting $\sqrt{a}=x,\sqrt{b}=y$, we have $x,y\in(0,1)$ and we must show that $\frac{x^2}{1-y^2}+\frac{y^2}{1-x^2}\geq\frac{2xy}{1-xy}$, which follows from Bergstrom's inequality:

$LHS=\frac{x^2}{1-y^2}+\frac{y^2}{1-x^2}\geq\frac{(x+y)^2}{(1-y^2)+(1-x^2)}=\frac{x^2+y^2+2xy}{2-x^2-y^2}\overset{(1)}{\geq}\frac{2xy}{1-xy}=RHS$, where (1) $\Leftrightarrow$

$\Leftrightarrow\frac{x^2+y^2+2xy}{2-x^2-y^2}\geq\frac{2xy}{1-xy}\Leftrightarrow(x-y)^2(xy+1)\geq0$, obviously with equality for $x=y$.

Let get back to the main problem. Using the Lemma we obtain:

$$\frac{a}{1-b}+\frac{b}{1-a}\geq\frac{2\sqrt{ab}}{1-\sqrt{ab}},\frac{b}{1-c}+\frac{c}{1-b}\geq\frac{2\sqrt{bc}}{1-\sqrt{bc}},\frac{c}{1-a}+\frac{a}{1-c}\geq\frac{2\sqrt{ca}}{1-\sqrt{ca}}$$

Adding the 3 above inequalities we obtain the conclusion.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM − GM inequality, we have

$$\frac{b}{1-a}+\frac{a}{1-b}\geq2\sqrt{\frac{ab}{(1-a)(1-b)}}=2\sqrt{\frac{ab}{\left(1-\sqrt{ab}\right)^2-\left(\sqrt{a}-\sqrt{b}\right)^2}}\geq\frac{2\sqrt{ab}}{1-\sqrt{ab}}.$$

Similarly, we have

$$\frac{c}{1-b}+\frac{b}{1-c}\geq\frac{2\sqrt{bc}}{1-\sqrt{bc}}\ \text{ and }\ \frac{a}{1-c}+\frac{c}{1-a}\geq\frac{2\sqrt{ca}}{1-\sqrt{ca}}.$$

Adding these inequalities yields the desired result. Equality holds iff $a=b=c$.