JP.527 If $x, y, z > 0$ with $x + y + z = 1$ and $\lambda \geq 21$ then:

$$\frac{1}{x^3 + y^3 + z^3} + \frac{\lambda}{xy + yz + zx} \geq 3(\lambda + 3)$$

Proposed by Marin Chirciu – Romania

Solution 1 by proposer

We use pqr method.

We denote: $p = x + y + z = 1, q = xy + yz + xz, r = xyz.$

We have: $x^3 + y^3 + z^3 = p^3 - 3pq + 3r = 1 - 3q + 3r$

We write the inequality: $\frac{1}{1-3q+3r} + \frac{\lambda}{q} \geq 3(\lambda + 3).$

Because $3r = 3xyz = 3xyz(x + y + z) \leq (xy + yz + zx)^2 = q^2$, it suffices to prove that:

$$\frac{1}{1 - 3q + q^2} + \frac{\lambda}{q} \geq 3(\lambda + 3) \Leftrightarrow 3(\lambda + 3)q^3 - (10\lambda + 27)q^2 + (6\lambda + 8)q - \lambda \leq 0 \Leftrightarrow$$

$$(3q - 1)[(\lambda + 3)q^2 - (3\lambda + 8)q + \lambda] \leq 0$$

which follows from $q \leq \frac{1}{3}$, true form $1 = (x + y + z)^2 \geq 3(xy + yz + zx) = 3q.$

Equality holds if and only if $x = y = z = \frac{1}{3}.$

Remark: The best equality it's obtained for $\lambda = 21.$

Case $\lambda = 21$

If $x, y, z > 0$ with $x + y + z = 1$ then

$$\frac{1}{x^3 + y^3 + z^3} + \frac{21}{xy + yz + zx} \geq 72$$

Marin Chirciu

Solution

We use pqr method.

We denote: $p = x + y + z = 1, q = xy + yz + xz, r = xyz.$

We have: $x^3 + y^3 + z^3 = p^3 - 3pq + 3r = 1 - 3q + 3r.$

The inequality can be written: $\frac{1}{1-3q+3r} + \frac{21}{q} \geq 72.$

Because $3r = 3xyz = 3xyz(x + y + z) \le (xy + yz + zx)^2 = q^2$, it suffices to prove that:

$$\frac{1}{1-3q+q^2} + \frac{21}{q} \ge 72 \Leftrightarrow 72q^3 - 237q^2 + 134q - 21 \le 0 \Leftrightarrow (3q-1)^2(8q-21) \le 0,$$

which follows from $q \le \frac{1}{3}$, true from $1 = (x+y+z)^2 \ge 3(xy+yz+zx) = 3q$.

Equality holds if and only if $x = y = z = \frac{1}{3}$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $p := x + y + z = 1,\ \ q := xy + yz + zx,\ \ r := xyz$. Since

$$x^3 + y^3 + z^3 = p^3 - 3pq + 3r$$

then the given inequality can be rewritten as follows

$$\lambda\left(\frac{1}{q} - 3\right) \ge 9 - \frac{1}{1 - 3q + 3r}.$$

Since $(x+y+z)^2 \ge 3(xy+yz+zx)$, then $q \le \frac{1}{3}$.

And since we have $(xy+yz+zx)^2 \ge 3xyz(x+y+z)$, then

$3r \le q^2$, so it suffices to prove that

$$21\left(\frac{1}{q} - 3\right) \ge 9 - \frac{1}{1 - 3q + q^2} \Leftrightarrow \frac{(1-3q)^2(21-8q)}{q(1-3q+q^2)} \ge 0,$$

which is true for $q \le \frac{1}{3}$.

So the proof is complete. Equality holds iff $x = y = z = \frac{1}{3}$.