ROMANIAN MATHEMATICAL MAGAZINE

JP.528 If $a,b,c>0$ such that $a+b+c=3$ and $0\le\lambda\le\frac{1}{2}$ then

$$\frac{1}{a^3+\lambda}+\frac{1}{b^3+\lambda}+\frac{1}{c^3+\lambda}\ge\frac{3}{\lambda+1}$$

Proposed by Marin Chirciu – Romania

Solution 1 by proposer

We prove: Lemma: If $x>0$ and $0\le\lambda\le\frac{1}{2}$ then

$$\frac{1}{x^3+\lambda}\ge\frac{\lambda+4-3x}{(\lambda+1)^2}$$

Proof: We use Tangent Line Method.We consider the function $f:(0,\infty)\to\mathbb{R}, f(x)=\frac{1}{x^3+\lambda}$

We have $f(1)=\frac{1}{\lambda+1}$. Tangent's equation in point $x_0=1$ is $y-f(x_0)=f'(x_0)(x-x_0)$.

We have $f'(x)=\frac{-3x^2}{(x^3+\lambda)^2}, f'(1)=\frac{-3}{(\lambda+1)^2}$.

Tangent's equation in point $x_0=1$ is $y-\frac{1}{\lambda+1}=\frac{-3}{(\lambda+1)^2}(x-1)$.

We obtain the tangent's equation $y=\frac{\lambda+4}{(\lambda+1)^2}-\frac{3x}{(\lambda+1)^2}$. We prove that:

$f(x)=\frac{1}{x^3+\lambda}\ge\frac{\lambda+4}{(\lambda+1)^2}-\frac{3x}{(\lambda+1)^2}$. We have:

$\frac{1}{x^3+\lambda}\ge\frac{\lambda+4-3x}{(\lambda+1)^2}\Leftrightarrow(x-1)^2(3x^2+(2-\lambda)x+1-2\lambda)\ge 0$, which follows from the condition from the hypothesis $0\le\lambda\le\frac{1}{2}$. Let's get back to the main problem.

Using the Lemma we obtain:

$$LHS=\sum\frac{1}{a^3+\lambda}\ge\sum\frac{\lambda+4-3a}{(\lambda+1)^2}=\frac{3(\lambda+4)-3\sum a}{(\lambda+1)^2}=\frac{3(\lambda+4)-3\cdot 3}{(\lambda+1)^2}=\frac{3}{\lambda+1}=RHS.$$

Equality holds if and only if $(a,b,c)=(1,1,1)$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We will first prove the lemma that for all $a\in(0,1)$ and $\lambda\in\left[0,\frac{1}{2}\right]$, we have

$$\frac{1}{a^3+\lambda}\ge\frac{-3a+\lambda+4}{(\lambda+1)^2}.$$

The inequality is equivalent to

$$\frac{3a^4-(\lambda+4)a^3+3\lambda a+1-2\lambda}{(a^3+\lambda)(\lambda+1)^2}\geq 0 \text{ or } \frac{(a-1)^2[3a^2+(2-\lambda)a+1-2\lambda]}{(a^3+\lambda)(\lambda+1)^2}\geq 0,$$

which is true and the proof of lemma is complete. Using this lemma, we obtain

$$\frac{1}{a^3+\lambda}+\frac{1}{b^3+\lambda}+\frac{1}{c^3+\lambda}\geq\frac{-3a+\lambda+4}{(\lambda+1)^2}+\frac{-3b+\lambda+4}{(\lambda+1)^2}+\frac{-3c+\lambda+4}{(\lambda+1)^2}$$
$$=\frac{-3(a+b+c)+3(\lambda+4)}{(\lambda+1)^2}=\frac{3}{\lambda+1},$$

as desired. Equality holds iff $a=b=c=1$.

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\frac{1}{a^3+\lambda}-\frac{1}{1+\lambda}-\frac{3(1-a)}{(1+\lambda)^2}=\frac{(1-a)\left(1+a+a^2\right)}{(a^3+\lambda)(1+\lambda)}-\frac{3(1-a)}{(1+\lambda)^2}$$
$$=\frac{1-a}{1+\lambda}\cdot\left(\frac{1+a+a^2}{a^3+\lambda}-\frac{3}{1+\lambda}\right)=\frac{1-a}{1+\lambda}\cdot\frac{1+\lambda+a+a\lambda+a^2+a^2\lambda-3a^3-3\lambda}{(a^3+\lambda)(1+\lambda)}$$
$$=\frac{1-a}{1+\lambda}\cdot\frac{(1-a)\left(1+a+a^2\right)+a(1-a)(1+a)+a^2(1-a)+\lambda\left(a^2+a-2\right)}{(a^3+\lambda)(1+\lambda)}$$
$$=\frac{1-a}{1+\lambda}\cdot\frac{(1-a)\left(1+a+a^2\right)+a(1-a)(1+a)+a^2(1-a)-\lambda(1-a)(a+2)}{(a^3+\lambda)(1+\lambda)}$$
$$=\frac{(1-a)^2}{1+\lambda}\cdot\frac{1+a+a^2+a+a^2+a^2-\lambda(a+2)}{(a^3+\lambda)(1+\lambda)}\overset{0\leq\lambda\leq\frac{1}{2}}{\geq}$$
$$=\frac{(1-a)^2}{1+\lambda}\cdot\frac{1+2a+3a^2-\frac{1}{2}(a+2)}{(a^3+\lambda)(1+\lambda)}=\frac{(1-a)^2}{1+\lambda}\cdot\frac{\frac{3a}{2}+3a^2}{(a^3+\lambda)(1+\lambda)}\geq 0$$

$\because a>0$ and $0\leq\lambda\leq\frac{1}{2}\therefore\frac{1}{a^3+\lambda}\geq\frac{1}{1+\lambda}+\frac{3(1-a)}{(1+\lambda)^2}$ and analogs

$$\Rightarrow\frac{1}{a^3+\lambda}+\frac{1}{b^3+\lambda}+\frac{1}{c^3+\lambda}\geq\frac{3}{\lambda+1}+\frac{3}{(1+\lambda)^2}\cdot\left(3-\sum_{cyc}a\right)\overset{a+b+c=3}{=}\frac{3}{\lambda+1}$$

$\therefore\frac{1}{a^3+\lambda}+\frac{1}{b^3+\lambda}+\frac{1}{c^3+\lambda}\geq\frac{3}{\lambda+1}\ \forall\, a,b,c>0$ such that : $a+b+c=3$ and $0\leq\lambda\leq\frac{1}{2}$, " = " iff $a=b=c=1$ (QED)