JP.529 If $a,b,c>0; x\in\mathbb{R}$ then:

$$\frac{a}{(b\sin^2 x+c\cos^2 x)^3}+\frac{b}{(c\sin^2 x+b\cos^2 x)^3}+\frac{c}{(a\sin^2 x+c\cos^2 x)^3}\geq\frac{27}{(a+b+c)^2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer, Solution 2 and generalization by Marin Chirciu-Romania, Solution 3 by Tapas Das – India

Solution 1 by proposer

$$\sum_{cyc}\frac{a}{b\sin^2 x+c\cos^2 x}=\sum_{cyc}\frac{a^2}{ab\sin^2 x+ac\cos^2 x}\geq$$

$$\overset{BERGSTROM}{\geq}\frac{(a+b+c)^2}{(ab+bc+ca)(\sin^2 x+\cos^2 x)}=\frac{(a+b+c)^2}{ab+bc+ca}\geq 3$$

Because: $(a+b+c)^2\geq 3(ab+bc+ca)$

$$a^2+b^2+c^2\geq ab+bc+ca$$

$$(a-b)^2+(b-c)^2+(c-a)^2\geq 0$$

Hence:

$$\sum_{cyc}\frac{a}{b\sin^2 x+c\cos^2 x}\geq 3 \qquad (1)$$

$$\sum_{cyc}\frac{a}{(b\sin^2 x+c\cos^2 x)^3}=\sum_{cyc}\frac{\left(\frac{a}{b\sin^2 x+c\cos^2 x}\right)^3}{a^2}\geq$$

$$\overset{RADON}{\geq}\frac{1}{(a+b+c)^2}\cdot\left(\sum_{cyc}\frac{a}{b\sin^2 x+c\cos^2 x}\right)^3\overset{(1)}{\geq}\frac{27}{(a+b+c)^2}$$

Equality holds for: $a=b=c$.

Solution 2 by Marin Chirciu-Romania

$$\sum\frac{a}{(b\sin^2 x+c\cos^2 x)^3}\geq\frac{27}{(a+b+c)^2}$$

$$LHS=\sum\frac{a}{(b\sin^2 x+c\cos^2 x)^3}=\sum\frac{\left(\frac{a}{b\sin^2 x+c\cos^2 x}\right)^3}{a^2}\overset{Radon}{\geq}$$

$$\overset{Radon}{\geq} \frac{\left(\sum \frac{a}{b\sin^2 x + c\cos^2 x}\right)^3}{(\sum a)^2} \overset{(1)}{\geq} \frac{3^3}{(\sum a)^2} = \frac{27}{(a+b+c)^2} = RHS,$$

where (1) $\Leftrightarrow \sum \frac{a}{b\sin^2 x + c\cos^2 x} \geq 3$, which follows from

$$\sum \frac{a}{b\sin^2 x + c\cos^2 x} = \sum \frac{a^2}{ab\sin^2 x + ac\cos^2 x} \overset{CS}{\geq} \frac{(\sum a)^2}{\sum(ab\sin^2 x + ac\cos^2 x)} =$$

$$= \frac{(\sum a)^2}{\sum bc(\sin^2 x + \cos^2 x)} = \frac{(\sum a)^2}{\sum bc} \overset{SOS}{\geq} 3.$$

Equality holds if and only if $a = b = c$.

Remark: The problem can be developed.

If $a, b, c > 0, x \in \mathbb{R}$ and $n \in \mathbb{N}$ then

$$\sum \frac{a}{(b\sin^2 x + c\cos^2 x)^{n+1}} \geq \frac{3^{n+1}}{(a+b+c)^n}$$

Marin Chirciu – Romania

Solution:

$$LHS = \sum \frac{a}{(b\sin^2 x + c\cos^2 x)^{n+1}} = \sum \frac{\left(\frac{a}{b\sin^2 x + c\cos^2 x}\right)^{n+1}}{a^n} \overset{Radon}{\geq}$$

$$\overset{Radon}{\geq} \frac{\left(\sum \frac{a}{b\sin^2 x + c\cos^2 x}\right)^{n+1}}{(\sum a)^n} \overset{(1)}{\geq} \frac{3^{n+1}}{(\sum a)^n} = \frac{3^{n+1}}{(a+b+c)^n} = RHS,$$

where (1) $\Leftrightarrow \sum \frac{a}{b\sin^2 x + c\cos^2 x} \geq 3$, which follows from

$$\sum \frac{a}{b\sin^2 x + c\cos^2 x} = \sum \frac{a^2}{ab\sin^2 x + ac\cos^2 x} \overset{CS}{\geq} \frac{(\sum a)^2}{\sum(ab\sin^2 x + ac\cos^2 x)} =$$

$$= \frac{(\sum a)^2}{\sum bc\,(\sin^2 x + \cos^2 x)} = \frac{(\sum a)^2}{\sum bc} \overset{SOS}{\geq} 3.$$

Equality holds if and only if $a = b = c$.

Note: For $n = 2$ we obtain Problem JP.529 from RMM, Number 36, Spring 2025, propose by Daniel Sitaru, Romania

Solution 3 by Tapas Das – India

$$\frac{a}{(b\sin^2 x+c\cos^2 x)^3}+\frac{b}{(c\sin^2 x+a\cos^2 x)^3}+\frac{c}{(a\sin^2 x+b\cos^2 x)^3}=$$

$$=\frac{a^4}{(ab\sin^2 x+ac\cos^2 x)^3}+\frac{b^4}{(bc\sin^2 x+ab\cos^2 x)^3}+\frac{c^4}{(ac\sin^2 x+bc\cos^2 x)^3}$$

$$\overset{Radon}{\geq}\frac{(a+b+c)^4}{[(\sin^2 x+\cos^2 x)(ab+bc+ca)]^3}\geq\frac{(a+b+c)^4}{\left[\frac{(a+b+c)^2}{3}\right]^3}=\frac{27}{(a+b+c)^2}$$

[Note: $\therefore (2x)^2\geq 3\sum xy$**]**