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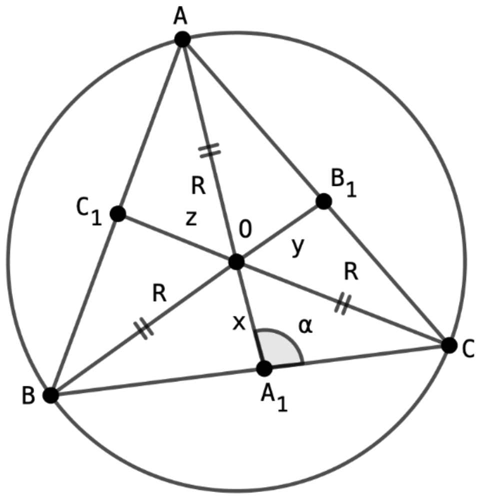
JP.530 In $\Delta ABC, O-$ circumcenter. A_1, B_1, C_1 are the intersection points of AO, BO, CO with BC, AC and AB respectively. R_1, R_2 and R_3 are circumradii of $\Delta BOC, \Delta AOC$ and ΔAOB respectively. Show that

$$R\left(\frac{1}{OA_1}+\frac{1}{OB_1}+\frac{1}{OC_1}\right)+3=\frac{4F}{R^2}\left(\frac{R_1}{BC}+\frac{R_2}{AC}+\frac{R_3}{AB}\right)$$

Proposed by Ertan Yildirim-Turkiye

Solution 1 by proposer, Solution 2 by Marin Chiricu – Romania

Solution 1 by proposer



$$\frac{F}{[BOC]}=\frac{\frac{1}{2}(R+x)\cdot a\cdot\sin\alpha}{\frac{1}{2}\cdot x\cdot a\cdot\sin\alpha}=\frac{R+x}{x}=\frac{R}{x}+1$$

$$\frac{\frac{abc}{4R}}{R^2\cdot\frac{a}{4R_1}}=\frac{abc\cdot R_1}{a\cdot R^3}=\frac{R}{x}+1$$

$$\Rightarrow\frac{4F\cdot R\cdot R_1}{a\cdot R^3}=\frac{4F}{R^2}\cdot\frac{R_1}{a}=\frac{R}{x}+1$$

Similarly $\frac{4F}{R^2}\cdot\frac{R_2}{b}=\frac{R}{y}+1, \frac{4F}{R^2}\cdot\frac{R_3}{c}=\frac{R}{z}+1$

$$\frac{4F}{R^2}\cdot\left(\frac{R_1}{a}+\frac{R_2}{b}+\frac{R_3}{c}\right)=3+R\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

$$\frac{4F}{R^2}\cdot\left(\frac{R_1}{BC}+\frac{R_2}{AC}+\frac{R_3}{AB}\right)=3+R\left(\frac{1}{OA_1}+\frac{1}{OB_1}+\frac{1}{OC_1}\right)$$

Solution 2 by Marin Chirciu – Romania

We evaluate the left side member.

Let be D – the leg of the height from A and M – the left of the perpendicular from O on BC, d_a – the distance from O to BC.

We have $\Delta ADC \sim \Delta OMA_1 \Rightarrow \frac{AD}{OM}=\frac{AA_1}{OA_1}\Leftrightarrow \frac{h_a}{d_a}=\frac{R+OA_1}{OA_1}\Leftrightarrow \frac{h_a}{d_a}=\frac{R}{OA_1}=$

$$=\frac{R}{OA_1}+1\Leftrightarrow \frac{R}{OA_1}=\frac{h_a}{d_a}-1$$

We obtain:

$$LHS=R\left(\frac{1}{OA_1}+\frac{1}{OB_1}+\frac{1}{OC_1}\right)+3=\sum\left(\frac{h_a}{d_a}-1\right)=3=\sum\frac{h_a}{d_a}=\sum\frac{ah_a}{ad_a}=\sum\frac{2F}{ad_a}=2F\sum\frac{1}{ad_a}\quad (1)$$

We evaluate the right hand member.

$$R_1=\frac{OB\cdot OC\cdot BC}{4[BOC]}=\frac{R\cdot R\cdot a}{4\cdot\frac{a\cdot d_a}{2}}=\frac{R^2}{2d_a}\Rightarrow \frac{R_1}{BC}=\frac{\frac{R^2}{2d_a}}{a}=\frac{R^2}{2ad_a}$$

$$RHS=\frac{4F}{R^2}\left(\frac{R_1}{BC}+\frac{R_2}{CA}+\frac{R_3}{AB}\right)=\frac{4F}{R^2}\sum\frac{R_1}{BC}=\frac{4F}{R^2}\sum\frac{R^2}{2ad_a}=2F\sum\frac{1}{ad_a}\qquad (2)$$

From (1) and (2) we deduce the conclusion.