ROMANIAN MATHEMATICAL MAGAZINE

**JP.531 If $a,b,c>0$ and $abc=1$ then:**

$$\left(\frac{a}{2}+\frac{b}{c}\right)^3+\left(\frac{b}{2}+\frac{c}{a}\right)^3+\left(\frac{c}{2}+\frac{a}{b}\right)^3>3\sqrt[4]{18}$$

*Proposed by Khaled Abd Imouti-Syria*

*Solution 1 by proposer, Solution 2 and generalizations by Marin Chirciu – Romania*

*Solution 1 by proposer*

$$\left(a+\frac{2b}{c}\right)^3=a^3+3a^2\cdot\frac{3b}{c}+3a\cdot\frac{4b^2}{c^2}+\frac{8b^3}{c^3}$$

$$\left(a+\frac{2b}{c}\right)^3=a^3+6\frac{a^2b}{c}+12\frac{ab^2}{c^2}+\frac{8b^3}{c^3}$$

$$\left(a+\frac{2b}{c}\right)^3>4\sqrt[4]{\frac{6\cdot12\cdot8\cdot a^6\cdot b^6}{c^6}},\left(a+\frac{2b}{c}\right)^3>4\sqrt[4]{\frac{2^4\cdot18\cdot a^8\cdot b^8\cdot c^2}{c^8a^2b^2}}$$

$$\left(a+\frac{2b}{c}\right)^3>4\cdot2\sqrt[4]{\frac{18\,a^8b^8c^2}{c^8\cdot a^2\cdot b^2}},\left(a+\frac{2b}{c}\right)^3>8\cdot\frac{a^2b^2}{c^2}\sqrt[4]{\frac{18c^2}{a^2b^2}}$$

$$abc=1\Rightarrow c=\frac{1}{ab},\qquad c^2=\frac{1}{a^2b^2},\qquad \frac{c^2}{a^2b^2}=\frac{1}{a^4b^4}$$

$$\left(a+\frac{2b}{c}\right)^3>8\frac{a^2b^2}{c^2}\sqrt[4]{\frac{18}{a^4b^4}},\left(a+\frac{2b}{c}\right)^3>8\cdot\frac{a^2b^2}{c^2}\cdot\frac{\sqrt[4]{18}}{ab}$$

$$\left(a+\frac{2b}{c}\right)^3>8\sqrt[4]{18}\cdot\frac{ab}{c^2}$$

$$\left(a+\frac{2b}{c}\right)^3+\left(b+\frac{2c}{a}\right)^3+\left(c+\frac{2a}{b}\right)^3>8\sqrt[4]{18}\left[\frac{ab}{c^2}+\frac{bc}{a^2}+\frac{ca}{b^2}\right]$$

$$\geq 8\sqrt[4]{18}\cdot3\cdot\sqrt[3]{\frac{a^2b^2c^2}{a^2b^2c^2}}$$

$$\left(a+\frac{2b}{c}\right)^3+\left(b+\frac{2c}{a}\right)^3+\left(c+\frac{2a}{b}\right)^3>24\sqrt[4]{18}$$

$$\left(\frac{a}{2}+\frac{b}{c}\right)^3+\left(\frac{b}{2}+\frac{c}{a}\right)^3+\left(\frac{c}{2}+\frac{a}{b}\right)^3>3\sqrt[4]{18}$$

*Solution 2 by Marin Chirciu – Romania*

$$\sum\left(\frac{a}{2}+\frac{b}{c}\right)^3\geq 3\sqrt[4]{18}$$

$$LHS=\sum\left(\frac{a}{2}+\frac{b}{c}\right)^3\overset{Holder}{\geq}\frac{\left[\sum\left(\frac{a}{b}+\frac{b}{c}\right)\right]^3}{9}=$$

$$=\frac{\left(\frac{a+b+c}{2}+\frac{b}{c}+\frac{c}{a}+\frac{a}{b}\right)^3}{9}\overset{AM-GM}{\geq}$$

$$\overset{AM-GM}{\geq}\frac{\left(\frac{3\sqrt[3]{abc}}{2}+3\sqrt[3]{\frac{b}{c}\cdot\frac{c}{a}\cdot\frac{a}{b}}\right)^3}{9}=\frac{\left(\frac{3}{2}+3\right)^3}{9}=\frac{\left(\frac{9}{2}\right)^3}{9}=\frac{9^2}{2^3}=$$

$$=\frac{81}{8}>3\sqrt[4]{18}=RHS$$

**The inequality is a strict one.**

**Remark: The problem can be developed.**

**If $a,b,c>0,abc=1$ and $n\in\mathbb{N}$ then:**

$$\sum\left(\frac{a}{2}+\frac{b}{c}\right)^n\geq 3\left(\frac{3}{2}\right)^n$$

*Marin Chirciu – Romania*

**Solution:**

$$LHS=\sum\left(\frac{a}{2}+\frac{b}{c}\right)^n\overset{Holder}{\geq}\frac{\left[\sum\left(\frac{a}{2}+\frac{b}{c}\right)\right]^n}{3^{n-1}}=$$

$$=\frac{\left(\frac{a+b+c}{2}+\frac{b}{c}+\frac{c}{a}+\frac{a}{b}\right)^n}{3^{n-1}}\overset{AM-GM}{\geq}$$

$$\overset{AM-GM}{\geq}\frac{\left(\frac{3\sqrt[3]{abc}}{2}+3\sqrt[3]{\frac{b}{c}\cdot\frac{c}{a}\cdot\frac{a}{b}}\right)^n}{3^{n-1}}=\frac{\left(\frac{3}{2}+3\right)^n}{3^{n-1}}=\frac{\left(\frac{9}{2}\right)^n}{3^{n-1}}=3\left(\frac{3}{2}\right)^n=RHS$$

**Equality holds if and only if $a=b=c$.**

**Note: For $n=3$ we obtain Problem JP.531 from RMM, Number 36, Spring 2025, proposed by Khaled Abd Imouti, Syria**

**Again the problem can be developed.**

**If $a,b,c>0, abc=1$ and $n\in\mathbb{N},\lambda>0$ then:**

$$\sum\left(\frac{a}{\lambda}+\frac{b}{c}\right)^n\geq 3\left(1+\frac{1}{\lambda}\right)^n$$

***Marin Chirciu – Romania***

**Solution:**

$$LHS=\sum\left(\frac{a}{\lambda}+\frac{b}{c}\right)^n\overset{Holder}{\geq}\frac{\left[\sum\left(\frac{a}{\lambda}+\frac{b}{c}\right)\right]^n}{3^{n-1}}=\frac{\left(\frac{a+b+c}{\lambda}+\frac{b}{c}+\frac{c}{a}+\frac{a}{b}\right)^n}{3^{n-1}}\overset{AM-GM}{\geq}$$

$$\overset{AM-GM}{\geq}\frac{\left(\frac{3\sqrt[3]{abc}}{\lambda}+3\sqrt[3]{\frac{b}{c}\cdot\frac{c}{a}\cdot\frac{a}{b}}\right)^n}{3^{n-1}}=\frac{\left(\frac{3}{\lambda}+3\right)^n}{3^{n-1}}=\frac{\left(\frac{3\lambda+3}{\lambda}\right)^n}{3^{n-1}}=$$

$$=3\left(\frac{3\lambda+3}{3\lambda}\right)^n=3\left(\frac{\lambda+1}{\lambda}\right)^n=RHS$$

**Equality holds if and only if $a=b=c$.**

**Note: For $n=3$ and $\lambda=2$ we obtain Problem JP.531 from RMM, Number 36, Spring 2025, proposed by Khaled Abd Imouti, Syria**