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JP.532 In any $\Delta ABC, I -$ incenter, $r -$ radii, $R -$ circumradii, $s -$ semiperimeter, the following relationship holds:

$$AB + BI + CI \leq 2(R + r)$$

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Solution 1 by proposer, Solution 2 by Tapas Das-India, Solution 3 by Marin Chirciu - Romania

Solution 1 by proposer

Firstly, we prove that

$$\frac{AI^2}{bc} + \frac{BC^2}{ca} + \frac{CI^2}{ab} = 1 \quad (1)$$

$$\sum_{cyc} \frac{AI^2}{bc} = \frac{1}{abc} \cdot \sum_{cyc} aAI^2 = \frac{1}{abc} \sum_{cyc} a \cdot \frac{r^2}{\sin^2 \frac{A}{2}} = \frac{r^2}{abc} \cdot \sum_{cyc} \frac{a}{\sin^2 \frac{A}{2}}$$

But, it is well-known that in any triangle ABC holds:

$$\sum_{cyc} \frac{a}{\sin^2 \frac{A}{2}} = \frac{4Rs}{r}$$

Hence,

$$\sum_{cyc} \frac{AI^2}{bc} = \frac{4Rrs}{abc} = \frac{4RF}{abc} = 1 \Rightarrow \text{(1) is true.}$$

Now, using Cauchy-Schwarz's inequality, we have:

$$\left(\frac{AI^2}{bc} + \frac{BC^2}{ca} + \frac{CI^2}{ab}\right)(ab + bc + ca) \geq (AI + BI + CI)^2 \quad (2)$$

From (1) and (2), it follows that

$$(AI + BI + CI)^2 \leq (ab + bc + ca) \quad (3)$$

But, $ab + bc + ca = s^2 + r^2 + 4Rr \quad (4)$

From (3) and (4) we get: $(AI + BI + CI)^2 \leq s^2 + r^2 + 4Rr$ (5) and

$$s^2 \leq 4R^2 + 4Rr + 3r^2 \text{ (Gerretsen)} \quad (6)$$

From (5) and (6), it follows that:

$$(AI + BI + CI)^2 \leq 4R^2 + 8Rr + 4r^2 \Leftrightarrow$$

$$(AI + BI + CI)^2 \leq 4R^2 + 8Rr + 4r^2 \Leftrightarrow AI + BI + CI \leq 2(R + r)$$

Solution 2 by Tapas Das-India

$$AI+BI+CI=r\csc\frac{A}{2}+r\csc\frac{B}{2}+r\csc\frac{C}{2}=$$

$$=r\sqrt{\frac{bc}{(s-b)(s-c)}}+r\sqrt{\frac{ca}{(s-c)(s-a)}}+r\sqrt{\frac{ab}{(s-a)(s-b)}}$$

$$\overset{CBS}{\leq} r\sqrt{\left(\sum bc\right)\cdot\sum\frac{1}{(s-b)(s-c)}}$$

$$=r\sqrt{(s^2+r^2+4Rr)\cdot\frac{s}{(s-a)(s-b)(s-c)}}$$

$$\overset{Gerretsen's}{\leq} r\sqrt{(4R^2+4Rr+3r^2+r^2+4Rr)\frac{1}{r^2}}=\sqrt{4(R^2+2Rr+r^2)}=2(R+r)$$

Solution 3 by Marin Chirciu – Romania

$$LHS=\sum AI=\sum\frac{r}{\sin\frac{A}{2}}\overset{(1)}{\leq} r\cdot 2\left(\frac{R}{r}+1\right)=2(R+r)=RHS,$$

where (1) $\Leftrightarrow \sum\frac{1}{\sin\frac{A}{2}}\leq 2\left(\frac{R}{r}+1\right)$, which follows from:

$$\sum\frac{1}{\sin\frac{A}{2}}=\sum\frac{1}{\sqrt{\frac{(s-b)(s-c)}{bc}}}=\sum\frac{\sqrt{bc}}{\sqrt{(s-b)(s-c)}}\overset{CBS}{\leq}$$

$$\overset{CBS}{\leq}\sqrt{\sum bc\sum\frac{1}{(s-b)(s-c)}}=$$

$$=\sqrt{(s^2+r^2+4Rr)\frac{1}{r^2}}=\frac{1}{r}\sqrt{s^2+r^2+4Rr}\overset{Gerretsen}{\leq}$$

$$\overset{Gerretsen}{\leq}\frac{1}{r}\sqrt{4R^2+4Rr+3r^2+r^2+4Rr}=$$

$$=\frac{1}{r}\sqrt{4R^2+8Rr+4r^2}=\frac{2}{r}\sqrt{R^2+2Rr+r^2}=\frac{2}{r}(R+r)=2\left(\frac{R}{r}+1\right)$$

Equality holds if and only if the triangle is equilateral.