JP.533 Let be the triangle ABC with $AD, BE, CF -$ altitudes and $H -$ orthocenter. Prove that:

$$\frac{HA}{HD}+\frac{HB}{HE}+\frac{HC}{HF}\ge 2\left(\left(\frac{R}{r}\right)^2-1\right)$$

Proposed by Marian Ursărescu – Romania

Solution 1 by proposer, Solution 2 by Tapas Das – India, Solution 3 by Marin Chirciu – Romania

Solution 1 by proposer

In any triangle ABC is well-known that:

$$\frac{HA}{HD}+\frac{HB}{HE}+\frac{HC}{HF}=\sum_{cyc}\tan A\tan B-3 \qquad (1)$$

and

$$\sum_{cyc}\tan A\tan B=\frac{s^2-r^2-4Rr}{s^2-(2R+r)^2}-3=$$

$$=\frac{s^2-r^2-4Rr-3s^2+12R^2+12Rr+3r^2}{s^2-(2R+r)^2}=\frac{12R^2+8Rr+2r^2-2r^2}{s^2-(2R+r)^2} \quad (3)$$

But $s^2\le 4R^2+4Rr+3r^2$ (Gerretsen) (4)

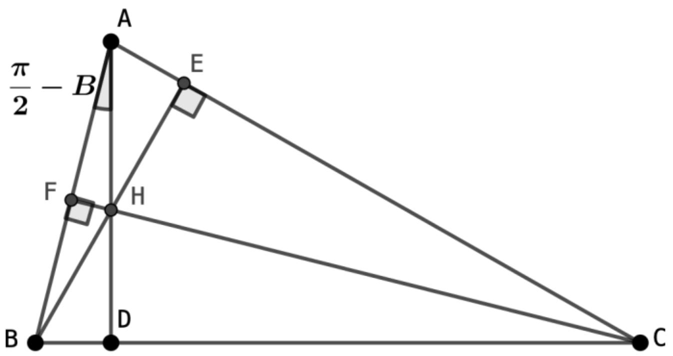
From (3) and (4), we get:

$$\frac{HA}{HD}+\frac{HB}{HE}+\frac{HC}{HF}\ge\frac{12R^2+8Rr+2r^2-8R^2-8Rr-6r^2}{4R^2+4Rr+3r^2-4R^2-4Rr-r^2}$$

Hence,

$$\frac{HA}{HD}+\frac{HB}{HE}+\frac{HC}{HF}\ge\frac{4R^2-4r^2}{2r^2}=\left(\left(\frac{R}{r}\right)^2-1\right)$$

Solution 2 by Tapas Das – India



In $\Delta ADB, \angle BAD = \frac{\pi}{2} - B$

In $\Delta AFC, AF = b\cos A$

In $\Delta AFH, \cos\left(\frac{\pi}{2} - B\right) = \frac{AF}{AH} = \frac{b\cos A}{AH} \Rightarrow AH = \frac{b\cos A}{\cos\left(\frac{\pi}{2}-B\right)} = \frac{b\cos A}{\sin B}$

$AH = 2R\cos A$ analog

In ΔAFH: $\tan\left(\frac{\pi}{2} - B\right) = \frac{FH}{AF} = \frac{FH}{b\cos A}$

$FH = \frac{b\cos A\cos B}{\sin B} = 2R\cos A\cos B$ (analog)

Now,

$$\frac{HA}{HD} + \frac{HB}{HE} + \frac{HC}{HF} = \frac{2R\cos A}{2R\cos B\cos C} + \frac{2R\cos B}{2R\cos A\cos C} + \frac{2R\cos C}{2R\cos A\cos B} =$$

$$= \frac{\sum\cos^2 A}{\cos A\cos B\cos C} = \frac{(\sum\cos A)^2 - 2\sum\cos A\cos B}{\cos A\cos B\cos C}$$

$$= \frac{\left(1+\frac{r}{R}\right)^2 - 2\frac{s^2+r^2-4R^2}{4R^2}}{\frac{s^2-(2R+r)^2}{4R^2}} = \frac{4(R+r)^2 - 2(s^2+r^2-4R^2)}{s^2-(2R+r)^2} =$$

$$= \frac{4(R^2+2Rr+r^2) - 2s^2 - 2r^2 + 8R^2}{s^2-(2R+r)^2} = \frac{12R^2+8Rr+2r^2-2s^2}{s^2-(4R^2+4Rr+r^2)} \geq$$

$$\overset{Gerretsen's}{\geq} \frac{12R^2+8Rr+2r^2-2(4R^2+4Rr+3r^2)}{4R^2+4Rr+3r^2-4R^2-4Rr-r^2}$$

$$= \frac{4R^2-4r^2}{2r^2} = 2\left(\frac{R}{r}\right)^2 - 2 = 2\left[\left(\frac{R}{r}\right)^2 - 1\right]$$

Solution 3 by Marin Chirciu – Romania

In acute ΔABC we have $HA = 2R\cos A$ and $HD = 2R\cos B\cos C$.

We obtain:

$$LHS = \sum\frac{HA}{HD} = \sum\frac{2R\cos A}{2R\cos B\cos C}\sum\frac{\cos A}{\cos B\cos C} = \frac{\sum\cos^2 A}{\prod\cos A} =$$

$$= \frac{\frac{6R^2+4Rr+r^2-s^2}{2R^2}}{\frac{s^2-(2R+r)^2}{4R^2}} = \frac{2(6R^2+4Rr+r^2-s^2)}{s^2-(2R+r)^2} \overset{(1)}{\geq} 2\left(\left(\frac{R}{r}\right)^2 - 1\right) = RHS$$

where (1) $\Leftrightarrow \frac{2(6R^2+4Rr+r^2-s^2)}{s^2-(2R+r)^2} \geq 2\left(\left(\frac{R}{r}\right)^2 - 1\right) \Leftrightarrow$

$$\Leftrightarrow \frac{(6R^2+4Rr+r^2-s^2)}{s^2-(2R+r)^2} \geq \frac{R^2-r^2}{r^2} \Leftrightarrow$$

$$\Leftrightarrow r^2(6R^2+4Rr+r^2-s^2) \geq (R^2-r^2)[s^2-(2R+r)^2] \Leftrightarrow$$

$$\Leftrightarrow R^2s^2 \leq (R^2-r^2)(2R+r)^2+r^2(6R^2+4Rr+r^2)$$

which follows from Gerretsen's inequality $s^2 \leq 4R^2+4Rr+3r^2$.

It remains to prove that:

$$R^2(4R^2+4Rr+3r^2) \leq (R^2-r^2)(2R+r)^2+r^2(6R^2+4Rr+r^2) \Leftrightarrow$$

$$\Leftrightarrow 3R^2r^2 \leq 3R^2r^2\text{, obviously.}$$

Equality holds if and only if the triangle is equilateral.