ROMANIAN MATHEMATICAL MAGAZINE

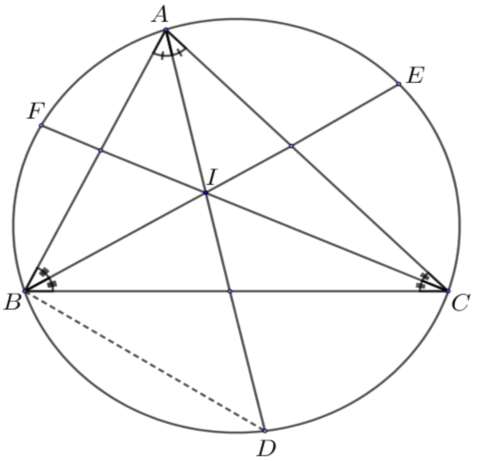
JP.534 In $\Delta ABC, I -$ incenter and D, E, F the points of contact of the cevians AI, BI, CI with the circumcircle, then the following relationship holds:

$$ID + IE + IF \le \frac{2(R^2 - Rr + r^2)}{r}$$

Proposed by Marian Ursărescu – Romania

Solution 1 by proposer, Solution 2 by Tapas Das – India, Solution 3 by Marin Chirciu – Romania

Solution 1 by proposer



In ΔIBC, from Law of sines, it follows:

$$\frac{ID}{\sin(IBD)} = \frac{IB}{\sin(IDB)} \quad (1)$$

But:

$$\sin(IBD) = \sin\left(\frac{B}{2} + \frac{A}{2}\right) = \sin\left(\frac{\pi}{2} - \frac{C}{2}\right) = \cos\frac{C}{2} \quad (2)$$

$$\sin(IDB) = \sin C \quad (3)$$

From (1), (2) and (3), we get:

$$ID = \frac{ID \cdot \cos\frac{C}{2}}{\sin C} = \frac{BI}{2\sin\frac{C}{2}}; \text{ and analogs } (4)$$

By adding, we have:

$$ID + IE + IF = \frac{1}{2}\left(\frac{BI}{\sin\frac{C}{2}} + \frac{CI}{\sin\frac{A}{2}} + \frac{AI}{\sin\frac{B}{2}}\right)$$

$$(ID+IE+IF)^2=\frac{1}{4}\left(\frac{BI}{\sin\frac{C}{2}}+\frac{CI}{\sin\frac{A}{2}}+\frac{AI}{\sin\frac{B}{2}}\right)^2 \quad (5)$$

$$\left(\frac{BI}{\sin\frac{C}{2}}+\frac{CI}{\sin\frac{A}{2}}+\frac{AI}{\sin\frac{B}{2}}\right)\overset{C-S}{\leq}(AI^2+BI^2+CI^2)\left(\frac{1}{\sin^2\frac{A}{2}}+\frac{1}{\sin^2\frac{B}{2}}+\frac{1}{\sin^2\frac{C}{2}}\right)=$$

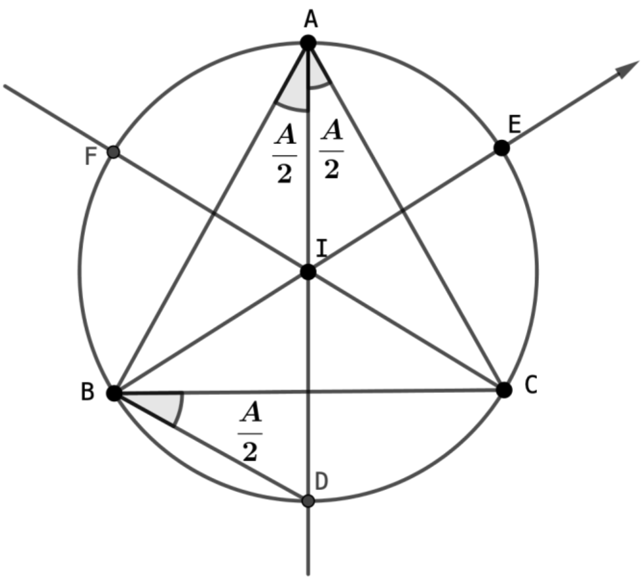
$$=(s^2+r^2-8Rr)\cdot\frac{s^2+r^2-8Rr}{r^2}=\frac{(s^2+r^2-8Rr)^2}{r^2} \quad (6)$$

From (5) and (6), it follows that:

$$(ID+IE+IF)^2\leq\frac{(s^2+r^2-8Rr)^2}{4r^2}$$

$$ID+IE+IF\leq\frac{s^2+r^2-8Rr}{2r}\overset{Gerretsen}{\leq}\frac{4R^2-4Rr+4r^2}{2r}=\frac{2(R^2-Rr+r^2)}{r}$$

Solution 2 by Tapas Das – India



$\angle DBC=\angle DAC=\frac{A}{2}$ **(angle on same one** DC**),** $\angle ACB=\angle ADB=C$

(angle on same one AB**). From** ΔABD **we have**

$$\frac{C}{\sin C}=\frac{BD}{\sin\frac{A}{2}}=2R\Rightarrow BD=2R\sin\frac{A}{2}$$

Now from ΔBID,

$$\angle IBD=\frac{B}{2}+\frac{A}{2}=\frac{\pi}{2}-\frac{C}{2}\quad(\because A+B+C=\pi),\ \angle BDI=C$$

$$\therefore \angle BID = \pi - \left(\frac{\pi}{2} - \frac{C}{2} + C\right) = \frac{\pi}{2} - \frac{C}{2}$$

$$\therefore \angle BID = \angle DBI \quad \therefore BD = ID$$

$$\therefore ID = 2R\sin\tfrac{A}{2} \quad \text{(analog)}$$

$$\therefore ID + IE + IF = 2R\sin\frac{A}{2} + 2R\sin\frac{B}{2} + 2R\sin\frac{C}{2}$$

$$= 2R\left(\sum \sin\frac{A}{2}\right) \overset{Jensen's}{\leq} 2R\,3\sin\left(\frac{A+B+C}{6}\right) = 2R\cdot 3\sin\frac{\pi}{6} = 3R$$

[$\sin\frac{x}{2}$ is concave in $x \in (0,\pi)$]

We need to show

$3R \leq \frac{2(R^2-Rr+r^2)}{r} \Rightarrow 2R^2 - 5Rr + 2r^2 \geq 0 \Rightarrow (R-2r)(2R-r) \geq 0$ True (Euler)

Solution 3 by Marin Chirciu – Romania

Using the power of point I towards circumcircle we obtain

$IA \cdot ID = R^2 - OI^2$ and as $IA = \frac{r}{\sin\frac{A}{2}}, OI^2 = R^2 - 2R$ it follows $\frac{r}{\sin\frac{A}{2}} \cdot ID = 2Rr \Leftrightarrow$

$$\Leftrightarrow ID = 2R\sin\frac{A}{2}$$

We obtain:

$$LHS = \sum ID = \sum 2R\sin\frac{A}{2} = 2R\sum\sin\frac{A}{2} \leq 2R\cdot\frac{3}{2} = 3R \overset{(1)}{\leq} \frac{2(R^2-Rr+r^2)}{r}$$
$$= RHS,$$

where (1) $\Leftrightarrow 3R \leq \frac{2(R^2-Rr+r^2)}{r} \Leftrightarrow 2R^2 - 5Rr + 2r^2 \geq 0 \Leftrightarrow (R-2r)(2R-r) \geq 0$,

see $R \geq 2r$, (Euler).

Equality holds if and only if the triangle is equilateral.