ROMANIAN MATHEMATICAL MAGAZINE

JP.535 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{r_a^4+r_b^2r_c^2}{r_b^2+r_c^2}\ge s^2$$

Proposed by Marian Ursărescu – Romania

Solution 1 by proposer, Solution 2 by Marin Chirciu – Romania

Solution 1 by proposer

Lemma. If $x,y,z>0$ then:

$$\sum_{cyc}\frac{x^2+yz}{y+z}\ge x+y+z$$

Proof of Lemma. We have:

$$\sum_{cyc}\frac{x^2+yz}{y+z}\ge x+y+z\Leftrightarrow\sum_{cyc}\frac{x^2+xy+yz+zx}{y+z}\ge 2(x+y+z)$$

$$\sum_{cyc}\frac{(x+y)(x+z)}{y+z}\ge (x+y)+(y+z)+(z+x)$$

which is true, because for $x+y=m, x+z=n, y+z=p$, we have:

$$\sum_{cyc}\frac{mn}{p}\ge m+n+p\Leftrightarrow (mn)^2+(np)^2+(pm)^2\ge mnp(m+n+p)$$

Hence,

$$\sum\nolimits_{cyc}\frac{r_a^4+r_b^2r_c^2}{r_b^2+r_c^2}\ge r_a^2+r_b^2+r_c^2=(4R+r)^2-2s^2 \qquad (1)$$

But: $(4R+r)^2\ge 3s^2$; (Doucet); (2). From (1) and (2), it follows that

$$\sum_{cyc}\frac{r_a^4+r_b^2r_c^2}{r_b^2+r_c^2}\ge s^2$$

Solution 2 by Marin Chirciu – Romania

$$LHS=\sum\frac{r_a^4+r_b^2r_c^2}{r_b^2+r_c^2}\overset{CS}{\ge}\frac{(\sum r_a^2)^2+(\sum r_br_c)^2}{\sum(r_b^2+r_c^2)}=\frac{(\sum r_a^2)^2+(s^2)^2}{2\sum r_a^2}\overset{(1)}{\ge}s^2=RHS$$

where (1) $\Leftrightarrow\frac{\left(\sum r_a^2\right)^2+s^4}{2\sum r_a^2}\ge s^2\Leftrightarrow(\sum r_a^2)^2+s^4\ge 2s^2\sum r_a^2\Leftrightarrow(\sum r_a^2-s^2)^2\ge 0$, with

equality for $\sum r_a^2=s^2$.

Equality holds if and only if the triangle is equilateral.