JP.536. InΔABC the following relationship holds:

$$\frac{2R}{r} \geq \frac{(4R+r)^2}{s^2} + 1$$

Proposed by Alex Szoros – Romania

Solution 1 by proposer, Solution 2 by Marin Chirciu – Romania, Solution 3 by Tapas Das – India

Solution 1 by proposer

We will demonstrate beforehand that:

$$\left(\sum a\right)\left(\sum \frac{1}{a}\right) \geq \frac{2\sum a^2}{\sum ab} + 7 \ \ (\forall) a, b, c > 0 \qquad (1)$$

The inequality is equivalent to

$$\frac{(\sum a)(\sum ab)}{abc} \geq \frac{2(\sum a^2) + 7\sum ab}{\sum ab} \Leftrightarrow \left(\sum a\right)\left(\sum ab\right)^2 \geq 2abc\left(\sum a^2\right) + 7abc\sum ab \quad (2)$$

On the other hand

$$\left(\sum a\right)\left(\sum ab\right)^2 \geq 3abc\left(\sum a\right)^2$$

It suffices to show that:

$$3abc\left(\sum a\right)^2 \geq 2abc\left(\sum a^2\right) + 7abc\sum ab \qquad \Leftrightarrow$$

$$3abc\sum a^2 + 6abc\sum ab \geq 2abc\left(\sum a^2\right) + 7abc\sum ab \quad \Leftrightarrow abc\sum a^2 \geq abc\sum ab \ \Leftrightarrow$$

$$\sum a^2 \geq \sum ab$$

and inequality (1) is proved.

From inequality (1) it follows that:

$$\left(\sum r_a\right)\left(\sum \frac{1}{r_a}\right) \geq \frac{2\sum r_a^2}{\sum r_a r_b} + 7 \qquad (3)$$

$$(3) \Leftrightarrow \ \left(\sum r_a\right)\left(\sum \frac{1}{r_a}\right) \geq \frac{2(\sum r_a)^2}{\sum r_a r_b} + 3$$

Using identities

$$\sum r_a = 4R + r\,,\ \sum \frac{1}{r_a} = \frac{1}{r} \ \text{ and } \sum r_a r_b = s^2$$

relation (3) becomes

$$\frac{4R+r}{r} \geq \frac{2(4R+r)^2}{s^2} + 3 \qquad \Leftrightarrow$$

$$\frac{4R}{r} \geq \frac{2(4R+r)^2}{s^2} + 2 \quad \Leftrightarrow \quad \frac{2R}{r} \geq \frac{(4R+r)^2}{s^2} + 1$$

Solution 2 by Marin Chirciu – Romania

Using Gerretsen's inequality $s^2 \geq 16Rr - 5r^2 \geq \frac{r(4R+r)^2}{R+r}$ it suffices to prove that:

$$\frac{2R}{r} \geq \frac{(4R+r)^2}{\frac{r(4R+r)^2}{R+r}} + 1 \Leftrightarrow \frac{2R}{r} \geq \frac{R+r}{r} + 1 \Leftrightarrow R \geq 2r, \text{(Euler)}$$

Equality holds if and only if the triangle is equilateral.

Solution 3 by Tapas Das – India

$$\frac{2R}{r} \geq \frac{(4R+r)^2}{s^2} + 1 \text{ or } \frac{2R}{r} \cdot s^2 \geq (4R+r)^2 + s^2$$

$$\frac{2R}{r}(16Rr - sr^2) \overset{Gerretsen}{\geq} (4R+r)^2 + 4R^2 + 4Rr + 3r^2$$

$$\Rightarrow 32R^2 - 10Rr \geq 20R^2 + 12Rr + 4r^2 \Rightarrow 12R^2 - 22Rr - 4r^2 \geq 0$$

$$\text{or } 6R^2 - 11Rr - 2r^2 \geq 0 \text{ or } 6R^2 - 12Rr + Rr - 2r^2 \geq 0$$

$$\text{or } 6R(R-2r) + r(R-2r) \geq 0$$

True (Euler)